Self-dual metrics on toric 4-manifolds: Extending the Joyce construction

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Abstract

Toric geometry studies manifolds M^{2n} acted on effectively by a torus of half their dimension, T^n. Joyce shows that for such a 4-manifold sufficient conditions for a conformal class of metrics on the free part of the action to be self-dual can be given by a pair of linear ODEs and gives criteria for a metric in this class to extend to the degenerate orbits. Joyce and Calderbank-Pedersen use this result to find representatives which are scalar flat Kähler and self-dual Einstein respectively.

We review some results concerning the topology of toric manifolds and the construction of Joyce metrics. We then extend this construction to give explicit complete scalar-flat Kähler and self-dual Einstein metrics on manifolds of infinite topological type, and to find a new family of Joyce metrics on open submanifolds of toric spaces. We then give two applications of these extensions — first, to give a large family of scalar flat Kähler perturbations of the Ooguri-Vafa metric, and second to search for a toric scalar flat Kähler metric on a neighbourhood of the origin in C^2 whose restriction to an annulus on the degenerate hyperboloid $\{(z_1,\, z_2)|z_1 z_2 = 0\}$ is the cusp metric.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Hugh Norman Griffiths*)

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Contents

Chapter 1

Introduction

The field of toric geometry is concerned with the study of spaces with a large number of symmetries, that is, whose symmetry groups contain a torus. This field utilises several different areas of geometry, including algebraic, symplectic, complex and differential geometry, and the spaces in question may carry different geometric structures. The high degree of symmetry makes these spaces particularly tractable.

The aim of this thesis is to extend a construction due to Joyce [23] which exploits the symmetry of compact four dimensional manifolds with an effective torus action to reduce the equations for a metric to be self-dual to a pair of linear ODEs, which are then solved explicitly. This construction makes it possible to explicitly construct Kähler metrics of scalar curvature 0, as well as self-dual Einstein metrics with torus symmetries. By considering metrics defined only on open sets of such a space we extend the family of solutions, construct Kähler and Einstein metrics with infinite topological type, and find a second family of solutions with which we can perturb the known solutions.

We begin in chapter 2 by giving definitions of self-dual metrics and a brief indication of why these metrics are an interesting subject of study ([2], [25]), and review some of the key results by which self-dual metrics are constructed.

Chapter 3 reviews two constructions which allow us to construct algebraic and symplectic toric spaces from combinatorial data, and shows how in the algebraic case the blow-up operation may be expressed in terms of this data, following the exposition in [14] and [20]. We examine the topology of such spaces via a result of Orlik-Raymond [27], which classifies 4-manifolds with a smooth effective T^2 action up to homeomorphism. This result is used to relate the two constructions, showing how we may pass from one type of combinatorial data to the other.

In chapter 4 we summarise the Joyce construction [23], from which a self-dual conformal class on a dense set in the 4-manifold can be found from a solution of a pair of ODEs satisfying a positivity condition, and exhibit a family of solutions of these equations, also presented in [23], and recount asymptotic conditions on the solutions which then allows us to extend this metric to the remainder of the manifold [23].

Chapter 5 investigates specific representatives of these conformal classes which give us scalar flat Kähler and self-dual Einstein metrics, following the work of [4], [23] and [5]. We show how the family of 'basic solutions' given by Joyce has been extended to include convolutions [5], to give spaces which are not simply connected [23], and to give complete Einstein spaces of infinite topological type [6].

In chapter 6 we find new complete Kähler and Einstein metrics on manifolds of infinite topological type, by taking an infinite sum of the solutions found in chapter 5 and removing the singular points. By finding bounds for the asymptotic behaviour of the resulting solution and the conformal factors near these points we show that the resulting spaces are complete. The result for Einstein manifolds extends the results of [6], but by extending a potential differently we are able to remove one constraint and produce many new metrics.

Finally, in chapter 7 we find a new type of solution to the equations in chapter 4 by considering power series. We give conditions for the resulting power series to converge on a suitable open set, and for the positivity condition to be met for these new solutions to give new Joyce metrics. We then apply the added freedom these solutions give us to some applications: Firstly we demonstrate how the Ooguri-Vafa metric [19] can be expressed as a Joyce solution, and use this to find a large family of scalar-flat Kähler perturbations. Secondly we show how we can use these new solutions to prescribe the metric on the degenerate fibres of the toric space, and use this to find sequences of metrics approximating a cusp metric on a suitable region.

Chapter 2

Self-dual spaces and metrics

Before reviewing the method of Joyce for constructing toric self-dual spaces, we first look at self-dual metrics in general. Self-dual metrics are interesting for a number of reasons — the self-duality condition for connections turns out to give minima of the Yang-Mills functional [2], self-dual metrics appear in a number of physical applications, such as the study of non-linear gravitons [22], and perhaps most importantly, Penrose's twistor construction [25] gives such metrics a very rich structure, as well as providing a means for their study.

After giving the definition of a self-dual metric, we briefly review Penrose's twistor construction, which associates to a self-dual manifold a complex 3-manifold. These two sections are based on material from [25] and [2]. Finally we will give an account of a selection of results by which self-dual metrics have been constructed.

2.1 Definition

We begin by giving the definition of a self-dual metric. This is given in [25] and [2]. Given an oriented Riemannian manifold (M, g) of dimension n, we can define the *Hodge star operator* on k forms by

$$\begin{aligned} * : \Lambda^k(M) &\to \Lambda^{n-k}(M) \\ \eta \wedge *\omega &= \langle \eta, \omega \rangle d\,\mathrm{vol} \qquad\qquad \forall \eta \in \Lambda^k(M) \end{aligned}$$

where $d\,\mathrm{vol}$ is the volume form on M and $\langle \cdot, \cdot \rangle$ the inner product on n-forms induced by the metric.

A special case of this occurs for 2-forms on a 4-manifold, where this map is a involution of $\Lambda^2(M)$,

$$*: \Lambda^2(M) \to \Lambda^2(M), \qquad *^2 = \mathrm{id}\,.$$

This map has eigenvalues $+1$ and -1 and we can decompose the space of 2-forms into a direct sum of eigenspaces,

$$\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$$

where we call $\Lambda^2_+(M)$ the *self-dual forms* and $\Lambda^2_-(M)$ the *anti-self-dual* forms.

Of particular interest to geometry is the application of this decomposition to the Riemann curvature tensor, viewed as a map on 2-forms,

$$\mathcal{R} : \Lambda^2(M) \to \Lambda^2(M).$$

With respect to this decomposition we can then write $\mathcal{R}$ as a block matrix,

$$\mathcal{R} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with

$$\begin{aligned} A &: \Lambda^2_+(M) \to \Lambda^2_+(M), \qquad & B &: \Lambda^2_-(M) \to \Lambda^2_+(M) \\ C &: \Lambda^2_+(M) \to \Lambda^2_-(M), \qquad & D &: \Lambda^2_-(M) \to \Lambda^2_-(M). \end{aligned}$$

Since $\mathcal{R}$ is self-adjoint,

$$C = B^*,\ A = A^* \text{ and } D = D^*.$$

Of these, B is the traceless Ricci curvature and

$$s = 2\,\mathrm{Tr}(\mathcal{R}) = 4\,\mathrm{Tr}(A) = 4\,\mathrm{Tr}(C)$$

is the scalar curvature.

We define

$$\begin{aligned} W_+ &= A - \tfrac{1}{3}\,\mathrm{Tr}(A) = \Lambda^2_+(M) \to \Lambda^2_+(M) \\ W_- &= C - \tfrac{1}{3}\,\mathrm{Tr}(C) = \Lambda^2_-(M) \to \Lambda^2_-(M) \end{aligned}$$

to be the *self-dual Weyl curvature* and *anti-self-dual Weyl curvature* respectively.

Definition 2.1.1. *If the anti-self-dual Weyl curvature vanishes, $W_- = 0$, we say*

(M, g) *is a* self-dual metric.

In fact W_+ and W_- are conformally invariant, so we can also define self-duality for a conformal class of metrics, $(M, [g])$.

2.2 Self-dual metrics

Self-duality has many consequences for the geometry of a space, and has complex interactions with many geometric structures. We will be particularly interested in three types of metric, namely Kähler, hyperkähler and Einstein metrics. We define these here, and briefly discuss the consequences of each condition for the curvature tensor.

Definition 2.2.1. *Let* (M^{2n}, g, J, ω) *be a manifold, Riemannian metric, complex structure and symplectic form respectively, with*

$$g(JX, JY) = g(X, Y) \qquad \forall X, Y \in T_xM, \; x \in M$$

satisfying

$$g(X, Y) = \omega(X, JY) \qquad \forall X, Y \in T_xM, \; x \in M.$$

Then we say (M, g, J, ω) *is a* Kähler manifold. *In this case*

$$h = g - i\omega$$

is a Hermitian metric on M*, considered as a complex manifold, and we say* h *is the* Kähler form *of* (M, g, J, ω).

Derdziński notes [10] that the self-dual Weyl curvature of a Kähler 4-manifold is given by a particular 2-form multiplied by the scalar curvature, and in particular a Kähler 4-manifold is anti-self-dual if and only if its scalar curvature is zero.

Definition 2.2.2. *A Riemannian manifold* (M, g) *is* hyperkähler *if there are three complex structures* I*,* J *and* K *with*

$$I^2 = J^2 = K^2 = IJK = -\operatorname{id}$$

and g *is Kähler with respect to each. That is, there are symplectic forms* $\omega_I, \omega_J, \omega_K$ *such that* (M, g, J, ω_α) *is a Kähler manifold for* $\alpha = I, J, K$.

In [2] it is observed that such a metric must be Ricci flat, and Calderbank-Pedersen [4] note that any self-dual Ricci-flat metric is locally hyperkähler.

Definition 2.2.3. *A Riemannian metric (M, g) is* Einstein *if the Ricci tensor is proportional to the metric — that is, for some $\lambda \in \mathbb{R}$,*

$$\mathrm{Ric} = \lambda g.$$

This condition is equivalent to asking that the trace-free Ricci curvature vanishes. Then if M is a 4-manifold we can apply the decomposition of the previous section and, in the notation used there, this condition becomes $B = 0$.

We now give a few examples of metrics of the various types defined above:

Example 2.2.4.

- *Since the Weyl tensor is conformally invariant, any conformally flat metric is self-dual, as noted by [25].*
- *The standard metric on $\mathbb{C}^2$ is Kähler. Let $(x_1+iy_1,\ x_2+iy_2)$, coordinates on $\mathbb{C}^2$. Then the complex structure is*

$$J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j} \qquad j = 1,\ 2.$$

The standard metric is

$$g = dx_1^2 + dy_1^2 + dx_2^2 + dy_2^2$$

and the symplectic form

$$\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$

Then $(\mathbb{C}^2,\ g,\ J,\ \omega)$ is a Kähler manifold. In fact, by identifying $\mathbb{C}^2$ with $\mathbb{H}$ we can see this space is in fact hyperkähler, the other complex structures being

$$\begin{aligned} J_2(\frac{\partial}{\partial x_1}) &= \frac{\partial}{\partial x_2} \qquad & J_2(\frac{\partial}{\partial y_1}) &= \frac{\partial}{\partial y_2} \\ J_3(\frac{\partial}{\partial x_1}) &= \frac{\partial}{\partial y_2} \qquad & J_3(\frac{\partial}{\partial y_2}) &= \frac{\partial}{\partial y_1}. \end{aligned}$$

- *The Fubini-Study metric on $\mathbb{C}P^2$ (see [18] p31, for example), given on the chart $[1,\, w_1, \dots,\, w_n]$ by*

$$g_{FS} = \frac{1 + \sum_{j=1}^{n} d|w_j|^2}{1 + \sum_{j=1}^{n} |w_j|^2} - \frac{\left(\sum_{j=1}^{n} \overline{w_j} dw_j\right)\left(\sum_{k=1}^{n} w_k d\overline{w_k}^2\right)}{\left(1 + \sum_{j=1}^{n} |w_j|^2\right)^2}.$$

  This is Kähler with the standard complex structure, given by multiplying by i, and the symplectic form

$$\omega_{FS} = \frac{i}{2\pi}\left(\frac{\sum_{j=1}^{n} dw_j \wedge d\overline{w_j}}{1 + \sum_{j=1}^{n} |w_j|^2} - \frac{(\sum_{j=1}^{n} \overline{w_j} dw_j) \wedge (\sum_{k=1}^{n} w_k d\overline{w_k})}{(1 + \sum_{j=1}^{n} |w_j|^2)^2}\right).$$

  This metric is Einstein and self-dual (with the complex orientation)[25], but as noted above it cannot be anti-self-dual since its scalar curvature is non-zero.

- *Any hyperkähler metric is of course Kähler, and as we noted must be Ricci flat. In particular it is Einstein, and also since it is scalar flat and Kähler is self-dual. In section (2.4) we discuss the Gibbons-Hawking ansatz, which provides many examples of such metrics.*

2.3 The Penrose Construction

We briefly recap Penrose's construction of the twistor space of a self-dual manifold. This construction gives a correspondence between self-dual manifolds and a family of complex 3-manifolds, and allows us to recast many differential geometric problems on the self-dual manifold in terms of holomorphic data.

Given a Riemannian 4-manifold let $P = S(\Lambda^-)$, the bundle of anti-self-dual 2-forms of unit length. Let $(\omega,\, x)$ be a point in this bundle. We can use the metric to associate to ω a skew-adjoint endomorphism on the tangent space at x,

$$J_{(\omega,\, x)} : T_x M \to T_x M \qquad\qquad J^* = -J,$$

and since ω is anti-self-dual and has unit length,

$$J^2 = -\,\mathrm{id}\,.$$

That is, this gives us an almost complex structure on T_xM.

Using the Levi-Civita connection we can split the tangent bundle into horizontal and vertical spaces,

$$T_{(\omega,\,x)}P = T_xM \oplus TF.$$

We can build an almost complex structure on this space by, at each point $(\omega,\,x)$, taking the almost complex structure $J_{(\omega,\,x)}$ on the horizontal space, and on the vertical space the complex structure induced by identifying the fibre with $\mathbb{C}P^1$. This gives an almost complex structure on P, and this is integrable if and only if M is self-dual. We say that P is the *twistor space* of M.

We can deduce some extra information about this twistor space from this construction. Locally we can define two irreducible spinor bundles, V_- and V_+, with

$$\Lambda^{\pm} \cong S^2 V^{\pm}.$$

We can identify the twistor space P with $\mathbb{P}(V_-)$, the projectivised anti-self-dual spinor bundle. In particular this gives a bundle

$$V_- \setminus \{0\} \to \mathbb{P}(V_-) \cong P.$$

Considering this bundle tells us that each fibre $\pi^{-1}(x) \subseteq P$ is a holomorphically embedded copy of $\mathbb{C}P^1$ with normal bundle $H \oplus H$, where H is the dual of tautological line bundle. We also get a free anti-holomorphic involution $\tau : P \to P$ which preserves the fibres, given by identifying the fibre with $\mathbb{C}P^1$ and applying the antipodal map.

In fact, these observations are sufficient for a complex 3-manifold to be the twistor space of a self-dual manifold:

Theorem 2.3.1. *(Penrose, [28] [2]) If Q is a complex 3-manifold fibred by projective lines with normal bundle $H \oplus H$ (where H is the dual of the tautological line bundle), with a free anti-holomorphic involution which preserves the fibres, then Q is, up to holomorphic equivalence, the twistor space $P(M)$ of some self-dual manifold M.*

In particular, self-dual manifolds can be found by searching for twistor spaces.

2.4 Constructing self-dual metrics

There have been many approaches to constructing self-dual metrics. Atiyah-Hitchin-Singer [25] note several examples - any conformally flat manifold must be self-dual, as are $\mathbb{C}P^2$ with the Fubini-Study metric and any Ricci-flat Kähler manifold (with the opposite of its usual orientation), in particular this includes the metrics found by Yau on $K3$ surfaces.

One approach to finding self-dual metrics explicitly comes from a calculation of Gibbons-Hawking [16] which takes harmonic functions on $\mathbb{R}^3$ and constructs hyperkähler metrics invariant under an S^1 action. The construction goes as follows (where we follow the exposition of [19]): Take $V : U \to \mathbb{R}$ a harmonic positive function on an open set $U \subseteq \mathbb{R}^3$ in Euclidean space. Let $\alpha = 2\pi i * dV$ a 2-form on U. If $\frac{i}{2\pi}\alpha \in H^2(U, \mathbb{Z})$, an integral 2-form, let $\pi : X \to U$ be a principal S^1 bundle with curvature $\pi^*\alpha$, and connection 1-form θ where $d\theta = \pi^*\alpha$. Then

$$g_{GH} = V(dr^2 + du_3^2 + r^2 d\xi^2) + \frac{1}{V}\theta_0^2,$$

where $\theta_0 = \frac{\theta}{2\pi i}$, is a hyperkähler metric on X. It is noted in [4] that any flat toric self-dual Einstein manifold can locally be obtained from this ansatz.

An important result of this construction is the Ooguri-Vafa metric. We will make use of this metric in chapter 7, so we give its construction here in some detail. This metric is obtained by applying the Gibbons-Hawking ansatz to a periodic potential, and after quotienting by this periodicity, can be used as a model space for degenerations of elliptic fibrations. Since this potential is also invariant under rotations about an axis, the resulting metric is toric.

In particular the potential is

$$V(r,\, \xi,\, u_3) = \sum_{k \in \mathbb{Z}} \left(\frac{1}{\sqrt{r^2 + (u_3 - k)^2}} - \lambda_k \right)$$

where r, ξ, u_3 are cylindrical coordinates on $\mathbb{R}^3$, $a_0 = 0$ and

$$\lambda_k = \begin{cases} 0 & k = 0 \\ \frac{1}{|k|} & k \neq 0 \end{cases}$$

so that

$$*dV = \sum_{k\in\mathbb{Z}} \left(\frac{r^2}{(r^2+(u_3-k)^2)^{\frac{3}{2}}} d\eta\wedge d\xi - \frac{r(u_3-k)}{(r^2+(u_3-k)^2)^{\frac{3}{2}}} d\rho\wedge d\xi \right).$$

Then

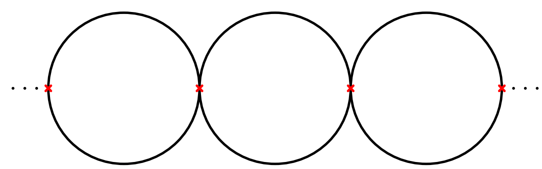
$$\theta_0 = \sum_{k\in\mathbb{Z}} \left(\frac{u_3-k}{\sqrt{r^2+(u_3-k)^2}} + \mathrm{sign}(k) \right) d\xi + \frac{1}{2\pi} dt \tag{2.1}$$

up to addition of a closed 1-form. From this we can calculate the Ooguri-Vafa metric,

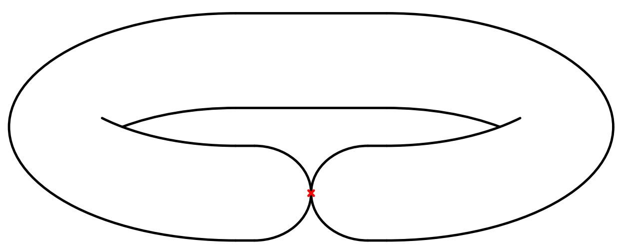
$$\begin{aligned} g_{OV} = V dr^2 + V du_3^2 + V r^2 d\xi^2 + \\ + \frac{1}{V} \left(\sum_{k\in\mathbb{Z}} \left(\frac{u_3-k}{\sqrt{r^2+(u_3-k)^2}} - \mathrm{sign}(k) \right) d\xi + \frac{1}{2\pi} dt \right)^2 \end{aligned} \tag{2.2}$$

on the appropriate S^1 bundle over $\mathbb{R}^3$, X.

Since the potential is invariant under rotation about the axis the metric is in fact toric. The non-free orbits form a bi-infinite chain of spheres, each meeting its neighbour in a single point. Translation by integers along the u_3-axis gives an additional $\mathbb{Z}$-action, and quotienting by this action reduces the chain of spheres to a ‘pinched’ torus.



(a) Degenerate orbits of X



(b) Degenerate orbits of $\frac{X}{\mathbb{Z}}$

Figure 2.1: The central fibre before and after quotienting by $\mathbb{Z}$.

We shall see in chapter 7 how seeing this calculation from a new direction will allow us to find many scalar flat Kähler perturbations of this metric.

In [22], Hitchin presents a method of constructing non-linear gravitons – that is, self-dual Einstein 4-manifolds which asymptotically have the topology of $\frac{S^3}{\Gamma} \times \mathbb{R}$ for finite groups Γ. Starting with Γ a cyclic group, this is done by considering the twistor space of $\frac{\mathbb{C}^2}{\Gamma}$ and constructing a desingularised version. The resulting self-dual manifold can be seen as a resolution of the singularity of $\frac{\mathbb{C}^2}{\Gamma}$. This method is extended in [8] to the case when Γ is a dihedral group, where the desingularisation is more challenging.

Poon, in [29], constructs a 1-parameter family of self-dual conformal metrics on $\mathbb{C}P^2 \# \mathbb{C}P^2$ with positive scalar curvature. This was achieved by using an observation of Hitchin to explicitly construct twistor spaces and using the twistor correspondence to recover the self-dual manifolds.

Donaldson-Friedman [11] consider the problem of finding self-dual metrics on connected sums of self-dual manifolds. They showed that the connected sum operation on self-dual manifolds can be expressed in twistor theoretic terms and used deformation theory to find the obstruction to finding a twistor space which is a small smoothing of the resulting space. They then show that when this obstruction vanishes the resulting twistor space does indeed recover the connected sum of the original spaces, with their original conformal metrics except at a small 'neck'. In particular this proves the existence of self-dual structures on $n\mathbb{C}P^2$, the n-fold connected sum of copies of $\mathbb{C}P^2$, and on connected sums of $\mathbb{C}P^2$s with $K3$ surfaces,

$$NK \# n\mathbb{C}P^2 \quad \forall N > 0,\, n \geq 2N + 1.$$

In [24], LeBrun generalises the Gibbons-Hawking ansatz [17] to give a new ansatz which constructs scalar flat Kähler metrics invariant under an S^1 action. He shows that every 4-manifold with a scalar flat Kähler metric invariant under an effective S^1 action arises in this way. In particular, applying this construction over $\mathcal{H}^3$ gives a large family of S^1-invariant self-dual metrics over $n\mathbb{C}P^2$.

Taubes [30] shows that there can be found an anti-self-dual metric on any smooth, compact oriented 4-manifold after connected summing with sufficiently many copies of $\overline{\mathbb{C}P^2}$, the complex projective plane with its orientation reversed (this is then self-dual in the opposite orientation). He constructs these metrics by observing that given a manifold M, taking a connected sum with $\overline{\mathbb{C}P^2}$ reduces the size of W_+ in some norm. Performing this operation iteratively, W_+ can be made arbitrarily small and a deformation theoretic argument is then used to find a perturbation of the resulting metric for which $W_+ = 0$.

2.5 The Joyce construction

Of particular interest to us will be a result of Joyce, which makes possible the discovery of a wide range of self-dual spaces in very explicit terms. We review some of the key results in the development of this approach here.

In [23] Joyce develops the approach of LeBrun further by searching for self-dual metrics with a T^2 action. He gives a pair of linear ODEs over $\mathcal{H}^2$, solutions of which (subject to a positivity condition) give self-dual metrics on $\mathcal{H}^2 \times T^2$. Finding a family of solutions of these equations and examining their behaviour close to the boundary then yields a large family of self-dual metrics on $n\mathbb{C}P^2$. This is the "Joyce construction" of the title. Joyce also observes that for each such conformal class of metrics there is a family of scalar flat Kähler representatives, parametrised by $\partial\mathcal{H}^2$. He shows that, while these spaces are all simply connected, it is possible to construct self-dual manifolds with fundamental group $\mathbb{Z}$ as a quotient of such a metric. We shall see these results in more detail in chapters 4 and 5.

In [4], Calderbank-Pedersen study self-dual Einstein manifolds with two commuting Killing fields. They give a condition for a Joyce conformal metric to possess an Einstein representative, and by relating these metrics to Einstein-Weyl spaces are able to show that any self-dual Einstein metric with two linearly independent Killing fields, and in particular any toric self-dual Einstein space, is either scalar flat and described by the Gibbons-Hawking ansatz, or is locally given by the Joyce construction. They further show that a large number of the Einstein metrics found by superposing the solutions found by Joyce correspond to metrics found by Galicki-Lawson's method [15] of taking quaternion-Kähler quotients of $\mathbb{H}P^{m-1}$.

In [5], Calderbank-Singer show how the solutions given by Joyce in [23] can be viewed as resolutions of quotient singularities by a series of blow-ups. They also reconsider Joyce's solutions in terms of distributions, and are able to construct infinite dimensional families of non-compact self-dual spaces with both scalar-flat Kähler and self-dual Einstein metrics.

They extend this approach in [6] to give a family of complete self-dual Einstein metrics on spaces of infinite topological type which can be thought of as the infinite analogue of the resolution of singularities in the previous paper. We will see the results of both of these papers in more detail in chapter 5, particularly in sections (5.3), (5.4) and (5.6), and give a generalisation of the latter result in (6.2).

In [7], Calderbank-Singer study toric self-dual Einstein metrics of positive scalar curvature with orbifold singularities. Many such metrics are provided by Galicki-Lawson [15] as quaternion quotients of $\mathbb{H}P^m$. Calderbank-Singer show by using the results of [4] that in fact any compact toric self-dual Einstein orbifolds of positive scalar curvature is, up to orbifold coverings, given by this construction.

Even more remarkably, in [13], Fujiki shows that any compact, connected, oriented 4-manifold M which is connected and simply connected and has a self-dual metric g invariant under a smooth effective T^2-action is diffeomorphic to a connected sum of copies of $\mathbb{C}P^2$ and the metric is one of those constructed by Joyce, proving a conjecture first put forth in [23].

This is proved by considering the twistor space Z of the 4-manifold, blown up along a number of subvarieties, to obtain a space $\hat{Z}$ and a holomorphic map $f : \hat{Z} \to P$, where P is a nonsingular rational curve, corresponding to taking a quotient by the torus action.

Considering this space locally about fibres over points in M on which the T^2-action is non-free, Fujiki builds a diffeomorphism invariant of the space and shows this invariant classifies such spaces. He then concludes by showing that the invariant corresponds to the diagram used by Orlik-Raymond [27] to describe the topology of toric 4-manifolds, and hence each diffeomorphism class has a representative amongst Joyce's metrics.

This approach is taken further by Wright [31], who shows that any compact 4-orbifold of positive orbifold Euler characteristic equipped with an anti-self-dual conformal structure whose isometry group contains a torus is also given by the Joyce construction.

Chapter 3

Toric Varieties and Torus Actions

Having considered self-dual spaces in general, we now consider toric spaces, along with some extra structures, namely algebraic structures and symplectic forms, which can be placed on them.

We explore a construction described by Fulton [14], which allows us to build a toric variety from a cone or system of cones, and give a few results which apply these methods to resolving singularities, from [5] and [14].

If our toric space is equipped with a symplectic structure which makes the torus action Hamiltonian, we will see that it can be described by a convex polytope using results from symplectic geometry [20] and the Atiyah-Guillemin-Sternberg convexity theorem ([1], [21]).

We demonstrate Orlik-Raymond's topological classification [27] of toric 4-manifolds, which, up to certain extra constraints, tells us that this combinatorial data is sufficient to classify these spaces in 4 dimensions.

Finally, we see how the combinatorial data given by each of these approaches is related.

3.1 Fans and toric varieties

This section follows the exposition of a construction by Fulton [14], which takes a cone or collection of cones over a lattice in a vector space to build an algebra, from which we obtain a toric variety via the Spec functor.

Definition 3.1.1. *A* toric variety *is a normal variety X containing an algebraic torus, $(\mathbb{C}^*)^n$ as a Zariski open subvariety.*

We will demonstrate how such a variety may be constructed from combinatorial data.

Definition 3.1.2. *A* rational cone $\sigma \subseteq \mathbb{R}^n$ *is the* $\mathbb{R}_+$ *cone generated by a finite collection of vectors in* $\mathbb{Z}^n$*,*

$$\sigma = \langle v_1, \ldots, v_k \rangle = \{\alpha_1 v_1 + \ldots + \alpha_k v_k \in \mathbb{R}^n | \alpha_1, \ldots, \alpha_k \geq 0\}, \qquad v_1, \ldots, v_k \in \mathbb{Z}^n.$$

Here, and throughout, $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$*.*

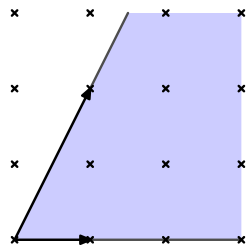


Figure 3.1: A rational cone.

Without loss of generality we assume the generators $v_1, \ldots, v_k$ of σ are *primitive*, that is, that they are elements of $\mathbb{Z}^n$ and are not multiples of any smaller vectors in $\mathbb{Z}^n$.

Of particular interest to us will be *simplicial* cones, which have a generating set consisting of a basis of $\mathbb{R}^n$.

We can then define the faces of a cone:

Definition 3.1.3. $\tau \subseteq \mathbb{R}^n$ *is a face of the cone* σ *if for some element of the dual lattice* $\phi_\tau \in (\mathbb{Z}^n)^* \subseteq (\mathbb{R}^n)^*$

$$\tau = \sigma \cap \ker \phi_\tau$$

with

$$\phi(\sigma) \subseteq \mathbb{R}_+.$$

We say ϕ_τ *is the* normal *to* τ*.*

That is, τ is the intersection of σ with a hyperplane which has a normal vector with integer coefficients, such that the cone lies entirely in one closed half-space. Each face is then a cone in its own right and is generated by the elements of the generating set lying in $\ker \phi$.

Lemma 3.1.4. *[14] If* σ *is a cone with faces* τ_1*,* τ_2*, then* $\tau_1 \cap \tau + 2$ *is also a face.*

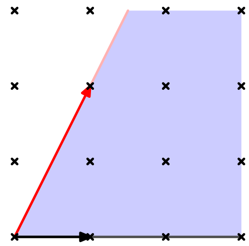


Figure 3.2: A face of a cone.

Proof. If τ_1, τ_2 are two faces with normals ϕ_{τ_1}, ϕ_{τ_2}, then $\phi_{\tau_1} + \phi_{\tau_2}$ is the normal to $\tau_1 \cap \tau_2$. $\square$

From each cone we can build a second cone in the dual space,

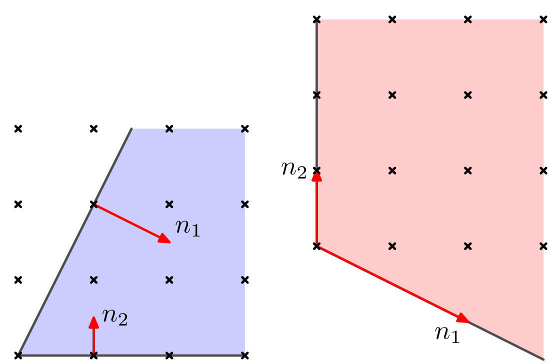
Definition 3.1.5. *The* dual cone, *$\check{\sigma}$, to a cone σ is the set of functionals under which the image of the cone lies in $\mathbb{R}_+$,*

$$\check{\sigma} = \{\phi \in (\mathbb{R}^n)^* | \phi(\sigma) \subseteq \mathbb{R}_+\}.$$

This is the convex hull of the rays

$$\{\lambda\phi_\tau | \lambda \in \mathbb{R}_+, \tau \text{ a co-dimension 1 face of } \sigma\}$$

and hence is a cone generated by the normals ϕ_τ to the faces of co-dimension 1.



(a) Normals to the faces (b) The dual cone

Figure 3.3: The dual cone.

Similarly the faces of the dual cone $\check{\sigma}$ correspond with the faces of σ, with each face taken to its annihilator. This identification reverses the order of inclusions.

Using the semigroup structure of the dual cone we can build a $\mathbb{C}$-algebra which we can consider as the coordinate ring of an affine variety.

Definition 3.1.6. *We define the algebra of a cone to be the vector space*

$$\mathbb{C}[\sigma] = \bigoplus_{u \in \check{\sigma} \cap \mathbb{Z}^n} \mathbb{C} \cdot X^u,$$

where $X^u = \prod_{i=1}^n X_i^{u_i} \in \mathbb{C}(X_1, \ldots, X_n)$, and multiplication satisfies

$$X^u \cdot X^v = X^{u+v}.$$

The semigroup $\check{\sigma} \cap \mathbb{Z}^n$ is generated by the intersection of $\mathbb{Z}^n$ with

$$\{\lambda_1 v_1 + \ldots + \lambda_k v_n | 0 \leq \lambda_i \leq 1\},$$

where $\{v_i\}$ are the generators of the cone $\check{\sigma}$. This set is finite, so the semigroup $\check{\sigma} \cap \mathbb{Z}^n$, and hence the algebra $\mathbb{C}[\sigma]$, are finitely generated.

We will abuse notation by writing

$$\mathbb{C}[P_1, \ldots, P_d] = \{P_1^{\alpha_1} \ldots P_d^{\alpha_d} | \alpha_1, \ldots, \alpha_d \geq 0\} \subseteq \mathbb{C}(X_1, \ldots, X_n)$$

for rational functions $P_1, \ldots, P_d$.

Definition 3.1.7. *$X(\sigma) = Spec(\mathbb{C}[\sigma])$ is then the variety corresponding to the cone σ.*

Each face of σ, as noted before, is itself a cone and hence gives a variety of its own. The inclusion of this face, $i : \tau \to \sigma$ induces an inclusion of the variety, $\hat{\imath} : X(\tau) \to X(\sigma)$. This gives us a complex of open subsets, partially ordered by inclusion. In particular the smallest face corresponds to an algebraic torus embedded in $X(\sigma)$ as an open subvariety (this smallest face is the vertex 0 unless σ contains a non-trivial vector space).

Example 3.1.8.

- $\mathbb{C}^n$. *Let σ_1 be the standard orthant in $\mathbb{R}^n$,*

$$\sigma_1 = \langle e_1, \ldots, e_n \rangle \subseteq \mathbb{R}^n.$$

The dual cone to this is the standard orthant in the dual space,

$$\check{\sigma}_1 = \langle \phi_1, \ldots, \phi_n \rangle \subseteq (\mathbb{R}^n)^*,$$

where $\phi_1, \ldots, \phi_n$ *are the standard dual basis. These are the generators of* $\check{\sigma}_1 \cap \mathbb{Z}^n$ *are just the standard dual basis vectors, so*

$$X(\sigma_1) = \mathrm{Spec}(\mathbb{C}[X^{\phi_1}, \ldots, X^{\phi_n}]) = \mathbb{C}^n.$$

- $(\mathbb{C}^*)^n$. *Now let* σ_2 *be the origin in* $\mathbb{R}^n$,

$$\sigma_2 = \{0\} \subseteq \mathbb{R}^n.$$

The dual cone now is the whole dual space,

$$\check{\sigma}_2 = \mathbb{R}^n = \langle \phi_1, -\phi_1, \ldots, \phi_n, -\phi_n \rangle.$$

Then $\check{\sigma}_2 \cap \mathbb{Z}^n = \mathbb{Z}^n$ *is spanned as a semigroup by the standard dual basis vectors and their inverses,*

$$\begin{aligned} \mathbb{C}[\sigma_2] &= \mathbb{C}[X^{\phi_1}, X^{-\phi_1}, \ldots, X^{\phi_n}, X^{-\phi_n}] \\ &= \mathbb{C}[X^{\phi_1}, (X^{\phi_1})^{-1}, \ldots, X^{\phi_n}, (X^{\phi_n})^{-1}] \cong \mathbb{C}(X_1, \ldots, X_n). \end{aligned}$$

This is the coordinate ring of the algebraic torus $(\mathbb{C}^*)^n$,

$$X(\sigma_2) = \mathrm{Spec}(\mathbb{C}[\sigma_2]) = (\mathbb{C}^*)^n.$$

- $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$. *Let*

$$\begin{aligned} \sigma_3 &= \langle e_1, \ldots, e_k, e_{k+1}, -e_{k+1}, \ldots, e_n, -e_n \rangle \\ &= \langle e_1, \ldots, e_k \rangle \times \langle e_{k+1}, -e_{k+1}, \ldots, e_n, -e_n \rangle. \end{aligned}$$

Each step in the process of building the toric variety is multiplicative over the Cartesian product, so we will find

$$X(\sigma_3) = X(\langle e_1, \ldots, e_k \rangle) \times X(\langle e_{k+1}, -e_{k+1}, \ldots, e_n, -e_n \rangle).$$

The dual cone is

$$\check{\sigma}_3 = \langle \phi_1, \ldots, \phi_k \rangle \times \langle \phi_{k+1}, -\phi_{k+1}, \ldots, \phi_n, -\phi_n \rangle,$$

so $\check{\sigma}_3 \cap \mathbb{Z}^n$ is spanned by the standard basis vectors and the inverses of the last $n-k$,

$$\mathbb{C}[\sigma_3] = \mathbb{C}[X^{\phi_1}, \ldots, X^{\phi_k}] \times \mathbb{C}(X^{\phi_{k+1}}, \ldots, X^{\phi_n}).$$

Hence the variety of σ_3 is, as expected,

$$X(\sigma_3) = \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}.$$

Proposition 3.1.9. *[14] If σ is a cone and $\tau \subseteq \sigma$ a face, the inclusion $i : \tau \to \sigma$ induces an inclusion*

$$\hat{\imath} : X(\tau) \to X(\sigma)$$

as an open set.

Proof. The face $\tau = \sigma \cap \ker \phi_\tau$ with ϕ_τ an element of the dual lattice. The dual cone $\check{\tau}$ is then generated by $\check{\sigma}$ and $\{-\phi_\tau\}$. The resulting algebra is then the extension of $\mathbb{C}[\sigma]$ by $(X^{\phi_\tau})^{-1}$. Hence we can naturally embed $\hat{\imath} : X(\tau) \to X(\sigma)$ as the set of points on which X^{ϕ_τ} is non-zero. $\square$

Example 3.1.10. *Consider the standard orthant, and one of its faces of codimension 1,*

$$\begin{aligned} \sigma &= \langle e_1, \ldots, e_n \rangle \subseteq \mathbb{R}^n \\ \tau &= \langle e_1, \ldots, e_{n-1} \rangle \subseteq \mathbb{R}^n. \end{aligned}$$

We have seen that the corresponding varieties are $\mathbb{C}^n$ and $\mathbb{C}^{n-1} \times \mathbb{C}^*$ respectively. Now the inclusion of the coordinate ring,

$$\mathbb{C}[\sigma] = \mathbb{C}[X^{\phi_1}, \ldots, X^{\phi_n}] \subseteq \mathbb{C}[\tau] = \mathbb{C}[X^{\phi_1}, \ldots, X^{\phi_{n-1}}, X^{\phi_n}, X^{-\phi_n}],$$

allows us to identify $X[\tau]$ *with the subset* $\{X^{\phi_n} \neq 0\}$ *of* $\mathbb{C}[\sigma]$,

$$\hat{\imath} \; : \; \mathbb{C}^{n-1} \times \mathbb{C}^* \to \mathbb{C}^n$$
$$\hat{\imath}(x_1, \dots, x_n) = (x_1, \dots, x_n).$$

Proposition 3.1.11. *[14] Let* σ *be a rational cone and consider the subvarieties given by its faces. The variety corresponding to the face of lowest dimension is an algebraic torus,* $(\mathbb{C}^*)^{(n-k)}$*. In particular, any variety constructed by this method is toric.*

Proof. First we must separate out any linear subspaces of σ. Let

$$V = \sigma \cap -\sigma,$$

the largest vector space contained in σ, and choose a basis of $\mathbb{Z}^n$ such that

$$\sigma = V \times \sigma_0 = \langle e_1, -e_1, \dots, e_k, -e_k, v_1, \dots, v_j \rangle$$

where σ_0 is a cone of smaller dimension and contains no non-trivial vector spaces.

Because the v_i are linearly independent, for each i there is a face of σ_0 which does not include v_i. Then if we take the sum of the functionals corresponding to the faces,

$$\phi = \sum_{\tau \text{ a face of } \sigma_0} \phi_\tau$$

then the kernel of this functional does not include any of the v_i, so the associated face is

$$\tau_0 = \ker \phi \cap \sigma = \mathbb{R}^k \times \{(0, \dots, 0)\}$$

and this face is minimal. Then

$$\tau_0 = \mathbb{R}^k \times \{(0, \dots, 0)\} = \langle e_1, -e_1, \dots, e_k, -e_k \rangle$$

and the dual cone is

$$\check{\tau}_0 = \langle e_{k+1}, -e_{k+1}, \dots, e_n, -e_n \rangle.$$

As we saw in the preceding example, this dual cone yields the variety

$$X(\tau_0) = (\mathbb{C}^*)^{n-k}.$$

□

So far we have constructed affine toric varieties — this approach can be extended to build more complex spaces using these affine pieces as charts. In order to do this we define a fan, a collection of cones meeting along faces. The cones give us a collection of charts and the shared faces will yield gluing maps between them.

Definition 3.1.12. *A* fan Δ *is a finite collection of rational cones in* $\mathbb{R}^n$*, none of which contain any non-trivial subspaces, such that if* $\sigma_1, \sigma_2 \in \Delta$ *then* $\sigma_1 \cap \sigma_2 \in \Delta$ *and is a face of both, and if* τ *is a face of* $\sigma \in \Delta$ *then* τ *is also in the fan,* $\tau \in \Delta$*.*

To simplify notation, when we specify the elements of a fan, we will assume that all faces are also included.

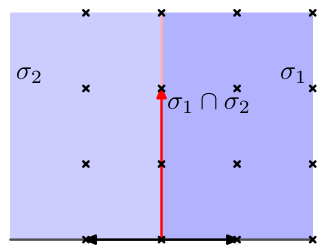


Figure 3.4: A fan, showing the intersection of two cones.

Definition 3.1.13. *Let* $X(\Delta) = \frac{\coprod_{\sigma\in\Delta} X(\sigma)}{\sim}$*, where if* $\tau \subseteq \sigma$*,*

$$x \in X(\tau) \sim y \in X(\sigma) \text{ if } i(x) = i(y),$$

where $i : X(\tau) \to X(\sigma)$ *the embedding map as above. This generates an equivalence relation* $\sim$*. The result is a variety since it is made up of affine varieties glued according to regular maps. It is also toric, since* $X(\{0\}) = (\mathbb{C}^*)^n$ *is embedded into each chart as an open set.*

Example 3.1.14. *A simple example of this construction is* $\mathbb{P}^n$ *— let*

$$\begin{aligned} \sigma_0 &= \langle e_1, \ldots, e_n \rangle \\ \sigma_i &= \langle e_1, \ldots, \hat{e}_i, \ldots, e_n, -e_1 - \ldots - e_n \rangle \quad 0 < i \leq n \end{aligned} \tag{3.1}$$

(here, and throughout, the hat denotes a member of the list which is omitted) and

$$\Delta = \{\sigma_i | 0 \leq i \leq n\}$$

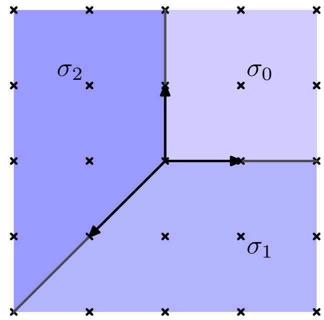


Figure 3.5: The fan of $\mathbb{P}^n$.

The dual cones are

$$\begin{aligned} \check{\sigma}_0 &= \langle \phi_1, \ldots, \phi_n \rangle \\ \check{\sigma}_i &= \langle -\phi_i,\, \phi_1 - \phi_i, \ldots,\, \phi_n - \phi_i \rangle \quad 0 < i \leq n, \end{aligned}$$

where $\phi_1, \ldots,\, \phi_n$ *are the standard basis of* $(\mathbb{R}^n)^*$, *and calculating the bases of* $\sigma_i \cap (\mathbb{Z}^n)^*$ *as a semigroup gives us the algebra*

$$\begin{aligned} \mathbb{C}[\sigma_0] &= \mathbb{C}[X^{\phi_1}, \ldots,\, X^{\phi_n}] \\ \mathbb{C}[\sigma_i] &= \mathbb{C}[X^{\phi_1 - \phi_i}, \ldots, X^{-\phi_i}, \ldots,\, X^{\phi_n - \phi_i}] \quad 0 < i \leq n \end{aligned}$$

since each of these is a polynomial algebra each chart is $\mathbb{C}^n$ *and the identifications are as follows:*

$$\begin{aligned} \Phi_{0,\,i} \;&:\; X[\sigma_0] \setminus \{z_i = 0\} \to X[\sigma_i] \setminus \{z_1 = 0\} \\ &\Phi_{0,\,i}(z_1, \ldots,\, z_n) = \left(\frac{z_1}{z_i}, \ldots,\, \frac{1}{z_i},\, \ldots,\, \frac{z_n}{z_i} \right) \\ \Phi_{i,\,j} \;&:\; X[\sigma_i] \setminus \{w_j = 0\} \to X[\sigma_j] \setminus \{w_i = 0\} \\ &\Phi_{i,\,j}(w_1, \ldots,\, w_n) = \left(\frac{w_1}{w_j}, \ldots,\, \frac{w_i}{w_j}, \ldots,\, \frac{1}{w_j}, \ldots,\, \frac{w_n}{w_j} \right) \end{aligned}$$

To see that the resulting space is $\mathbb{P}^n$ *we map the charts into projective space as follows:*

$$\begin{aligned} \Psi_0 \;&:\; X[\sigma_0] \to \mathbb{P}^n, \qquad \Psi_0(z_1, \ldots,\, z_n) = [1,\, z_1, \ldots,\, z_n] \\ \Psi_i \;&:\; X[\sigma_i] \to \mathbb{P}^n, \qquad \Psi_i(w_1, \ldots,\, w_n) = \left[\frac{1}{w_i},\, \frac{w_1}{w_i}, \ldots,\, 1, \ldots,\, \frac{w_n}{w_i} \right] \end{aligned}$$

These identifications are well defined with respect to the gluing operations and so they give a bijection from $X(\Delta)$ *to* $\mathbb{P}^n$.

Having seen how we may build a toric variety from a fan, we will want to use this fan to deduce properties of the variety. In particular we will give a simple criterion for the variety to be non-singular — namely that the generators of each maximal cone in its fan be a basis of $\mathbb{Z}^n$. We then show, as an example of the kind of results we can obtain from this approach, that for a non-singular variety we may also deduce the fundamental group. These results are presented in [14].

Theorem 3.1.15. *[14] A simplicial cone $\sigma = \langle v_1, \dots, v_n \rangle$ corresponds to a non-singular toric variety $X(\sigma)$ if and only if $v_1, \dots, v_n$ form a basis of $\mathbb{Z}^n$.*

Proof. Since the construction is independent of the choice of basis, if $v_1, \dots, v_n$ are a $\mathbb{Z}$-basis we can assume without loss of generality that $\sigma = \langle e_1, \dots, e_n \rangle$, and we have seen that then $X(\sigma) = \mathbb{C}^n$.

Now suppose $v_1, \dots, v_n$ are not a $\mathbb{Z}$-basis. Let k be the smallest number such that $v_1, \dots, v_k$ are not a basis of $\mathbb{Z}^n \cap (\langle v_1, \dots, v_k \rangle \otimes \mathbb{R})$. Note that since the v_i are primitive $k \geq 2$, and since σ is simplicial that $\tau = \langle v_1, \dots, v_k \rangle$ is a face of σ.

Then $v_1, \dots, v_{k-1}$ are a basis of the set of lattice points inside the vector space they span, so after a change of basis,

$$\tau = \langle e_1, \dots, e_{k-1}, \lambda e_k - \alpha_1 e_1 - \dots - \alpha_{k-1} e_{k-1} \rangle \qquad \lambda \geq 2.$$

The dual cone of this face,

$$\tilde{\tau} = \langle \lambda\phi_1 + \alpha_1\phi_k, \dots, \lambda\phi_{k-1} + \alpha_{k-1}\phi_{k-1}, \phi_k, \phi_{k+1}, -\phi_{k+1}, \dots, \phi_n, -\phi_n \rangle,$$

where $\phi_1, \dots, \phi_n$ are the standard dual basis. Now, however, the generators of the dual cone do not generate $\tilde{\tau} \cap (\mathbb{Z}^n)^*$ as a semigroup,

$$\begin{aligned} \mathbb{C}[\tau] \quad = \quad & \mathbb{C}[\{X_1^{a_1} \dots X_{k-1}^{a_{k-1}} X_k^b X_{k+1}^{c_{k+1}} \dots X_n^{c_n} \mid \\ & \qquad a_1, \dots, a_{k-1}, b \in \mathbb{N}, \lambda b - \sum \alpha_i a_i \geq 0, c_{k+1}, \dots, c_n \in \mathbb{Z}\}]. \end{aligned}$$

Applying the substitution $U = X_k^{\frac{1}{\lambda}}, V_j = X_j^{\alpha_j} U^{\alpha_i}$,

$$\begin{aligned}
\mathbb{C}[\tau] &= \mathbb{C}\left[\{V_1^{a_1}\dots V_{k-1}^{a_{k-1}} U^{\lambda b - \sum \alpha_i a_i} X_{k+1}^{c_{k+1}} \dots X_n^{c_n} | \right. \\
&\qquad \left. a_1, \dots, a_{k-1}, b \in \mathbb{N}, c_{k+1}, \dots, c_n \in \mathbb{Z}\}\right] \\
&= \mathbb{C}\left[\{V_1^{a_1}\dots v_{k-1}^{a_{k-1}} U^d | a_1, \dots, a_{k-1}, d \in \mathbb{N}, \sum \alpha_i a_i + d \in \lambda\mathbb{N}\}\right] \times \\
&\qquad \times \mathbb{C}\left[\{X_{k+1}^{c_{k+1}} \dots X_n^{c_n} | c_1, \dots, c_n \in \mathbb{Z}\}\right].
\end{aligned}$$

Since the first factor is generated by those polynomials in $\mathbb{C}[V_1, \dots, V_{k-1}, U]$ with weight divisible by λ, with weighting

$$\text{weight}(U) = 1, \quad \text{weight}(V_i) = \alpha_i.$$

This factor consists of precisely those polynomials invariant under the $\mathbb{Z}_\lambda$ action

$$m \cdot (v_1, \dots, v_{k-1}, u) = (\omega^{m\alpha_1} v_1, \dots, \omega^{m\alpha_{k-1}} v_{k-1}, \omega^m u)$$

where ω is a λth root of unity, so its Spec is the quotient of $\mathbb{C}^k$ by this group action. Hence

$$X(\tau) = \frac{\mathbb{C}^k}{G} \times \mathbb{C}^{n-k}.$$

where G is the group generated by

$$\begin{pmatrix} \omega^{\alpha_1} & & \\ & \ddots & \\ & & \omega^{\alpha_n} \end{pmatrix}$$

□

Example 3.1.16. *The proof of this theorem yields a further example, that if*

$$\sigma = \langle e_1, \dots, e_{n-1}, \lambda e_n + \alpha_1 e_1 + \dots + +\alpha_{n-1} e_{n-1} \rangle$$

then

$$X(\sigma) = \frac{\mathbb{C}^n}{G}$$

where G is the group generated by

$$\begin{pmatrix} \omega^{\alpha_1} & & \\ & \ddots & \\ & & \omega^{\alpha_n} \end{pmatrix}$$

and ω is a λth root of unity. In particular this is an orbifold of degree λ.

Expressing a variety as a fan also tells us about its fundamental group, for example:

Theorem 3.1.17. *[14] If $X(\sigma)$ is the affine variety associated to a k-dimensional cone $\sigma \subseteq \mathbb{R}^n$, then $\pi_1(X(\sigma)) \cong \mathbb{Z}^{n-k}$.*

Proof. Suppose first that $k = n$. We use without proof that, since the maximal complex torus T^N is an open subvariety of $X(\sigma)$ and $X(\sigma)$ is a normal variety, the inclusion induces a surjection of fundamental groups,

$$\hat{\imath} : \pi_1(T^N) \to \pi_1(X(\sigma)).$$

Now, take $v \in (\mathbb{Z}^n)^*$ and consider the loop

$$\begin{aligned} \lambda_v \quad &: \quad S^1 \to T^N \\ &\quad\ \lambda_v(e^{i\phi}) = (e^{i\phi v_1}, \ldots, e^{i\phi v_n}). \end{aligned}$$

We can contract this loop as follows,

$$\begin{aligned} \tilde{\lambda}_v \quad &: \quad S^1 \times I \to X(\sigma) \\ &\quad\ \tilde{\lambda_v}(e^{i\phi}, r) = ((re^{i\phi})^{v_1}, \ldots, (re^{i\phi})^{v_n}) \end{aligned}$$

exactly when $\lim_{r\to 0}(\tilde{\lambda}_v(e^{i\phi}, r)) \in X(\sigma)$, so consider this limit.

Given $v \in \sigma$, let τ be the smallest face containing v. We can define an algebra homomorphism by

$$\begin{aligned} \chi_v \quad &: \quad \mathbb{C}[\sigma] \to \mathbb{C} \\ &\quad\ \chi_v(X^\phi) = \begin{cases} 0 & \text{otherwise} \\ 1 & \text{if } \phi \in \tau. \end{cases} \end{aligned}$$

The zeros of this homomorphism then form a maximal ideal and hence this corresponds to a point x_v in $X(\sigma) = \operatorname{Spec}\mathbb{C}[\sigma]$. This point is then the above limit,

$$x_v = \lim_{r\to 0}(\tilde{\lambda}_v(e^{i\phi}, r))$$

and hence λ_v is contractible whenever $v \in \sigma$. However, since σ spans $\mathbb{R}^n$ we can find a basis of such vectors in $\mathbb{Z}^n$, corresponding to a generating set for $\pi_1(T^N)$. Since the image of these loops is 0 and $\hat{\imath}$ is surjective,

$$\pi_1(X(\sigma)) = \operatorname{Im}\hat{\imath} = \{0\}.$$

Now for a general k, we can consider σ as a cone in $\mathbb{R}^k$, σ' and $X(\sigma) = X(\sigma')\times(\mathbb{C}^*)^{n-k}$. Then $X(\sigma')$ is simply connected and

$$\pi_1(X(\sigma)) = \pi_1((\mathbb{C}^*)^{n-k}) = \mathbb{Z}^{n-k}.$$

$\square$

Another construction which has a very elegant formulation in the combinatorial picture is that of blow-ups. We demonstrate how we may blow-up a point in a toric variety of this kind, as well as larger subvarieties. We also see how we may perform a weighted blow-up, in which the exceptional divisor is not $\mathbb{P}^n$ but a weighted projective space, $\mathbb{P}^n(\alpha_1, \ldots, \alpha_n)$.

We will use these characterisations to provide two methods for resolving different kinds of singularities — the first allows us to resolve a two dimensional orbifold singularity using a sequence of blow-ups, the second will allow us to resolve cyclic quotient singularities in any dimension using a series of weighted blow-ups.

Proposition 3.1.18. *[14] Let $\sigma = \langle v_1, \ldots, v_n\rangle$ be a simplicial, non-singular cone. Then the fan Δ consisting of cones*

$$\sigma_j = \langle v_1, \ldots, \hat{v_j}, \ldots, v_n, v_1 + v_2 + \ldots + v_n\rangle \quad j \geq 1$$

has variety $X(\Delta)$, the blow-up of $\mathbb{C}^n$ at 0.

Proof. After a change of basis, $\sigma = \langle e_1, \ldots, e_n\rangle$, so it is sufficient to calculate the

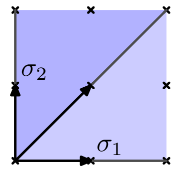


Figure 3.6: The fan Δ.

varieties in this case. Let

$$\sigma_j = \langle e_1, \ldots, \hat{e_j}, \ldots, e_n, e_1 + \ldots + e_n \rangle.$$

Then

$$\begin{aligned} \check{\sigma}_j &= \langle \phi_j, \phi_1 - \phi_j, \ldots, \phi_n - \phi_j \rangle \\ \mathbb{C}[\sigma_j] &= \mathbb{C}[X^{\phi_j}, X^{\phi_1 - \phi_j}, \ldots, X^{\phi_n - \phi_j}]. \end{aligned}$$

The gluing maps induce homomorphisms between the coordinate rings (after suitable extensions):

$$\begin{aligned} \Phi_{i\,j} \;\;:\;\; & \mathbb{C}[\sigma_j][(X^{\phi_i - \phi_j})^{-1}] \to \mathbb{C}[\sigma_i][(X^{\phi_j - \phi_i})^{-1}] \\ & \Phi_{i\,j}(X^{\phi_j}) = X^{\phi_j - \phi_i} X^{\phi_i} \\ & \Phi_{i\,j}(X^{\phi_k - \phi_j}) = \frac{X^{\phi_k - \phi_i}}{X^{\phi_j - \phi_i}} \end{aligned}$$

This extends to an isomorphism of the coordinate rings, and hence induces a biregular isomorphism:

$$\begin{aligned} \Psi_{i\,j} \;\;:\;\; & X(\sigma_i) \setminus \{z_j = 0\} \to X(\sigma_j) \setminus \{z_i = 0\} \\ & \Psi_{i\,j}(z_0, \ldots, \hat{z_i}, \ldots, z_n) = \left(z_j z_0, \frac{z_1}{z_j}, \ldots, \frac{z_n}{z_j} \right). \end{aligned}$$

These maps then tell us how to construct the variety $X(\sigma_i)$. It remains to show that this variety is the blow-up of $\mathbb{C}^n$. Let

$$\hat{\mathbb{C}}^n = \{(z_1, \ldots, z_n, [w_1, \ldots, w_n]) \in \mathbb{C}^n \times \mathbb{P}^{n-1} | z_i w_j = z_j w_i \,\forall i, j \leq n\},$$

the blow-up of $\mathbb{C}^n$ at the origin. Then define maps

$$\begin{aligned}\Theta_j \quad &: \quad X[\sigma_j] \to \hat{\mathbb{C}^n} \\ &\Theta_j(z_0,\, z_1, \ldots,\, \hat{z_j}, \ldots,\, z_n) = \\ &\qquad = (z_0 z_1, \ldots, z_0, \ldots,\, z_0 z_n,\, [z_0, \ldots,\, 1, \ldots,\, z_n]).\end{aligned}$$

Note that these maps are invariant under the gluing maps and hence lift to give us a map on the full variety. This is bijective, so $X(\Delta)$ is biregular to $\hat{\mathbb{C}^n}$. □

Corollary 3.1.19. *[14] Let $\sigma = \langle v_1, \ldots,\, v_n\rangle$ be a simplicial, non-singular cone and consider the face $\tau = \langle v_1, \ldots,\, v_k\rangle$. The fan Δ consisting of the cones*

$$\sigma_j = \langle v_1, \ldots,\, \hat{v_j}, \ldots,\, v_k,\, v_1 + \ldots + v_k\rangle \times \langle v_{k+1}, \ldots,\, v_n\rangle \qquad j \leq k$$

gives a variety $X(\Delta)$, the blow-up of $\{0\} \times \mathbb{C}^{n-k}$.

Proof. After a change of basis we may assume $\sigma = \langle e_1, \ldots,\, e_n\rangle$. Now, consider cones

$$\tilde{\sigma_j} = \langle e_1, \ldots,\, \hat{e_j}, \ldots,\, e_k\rangle \subseteq \mathbb{R}^k.$$

We have seen that the fan of these cones gives the blow-up of $\mathbb{C}^k$ at 0.

$$\begin{aligned}\mathbb{C}[\sigma_j] \quad &= \quad \mathbb{C}[\tilde{\sigma}_j][X^{\phi_{k+1}}, \ldots,\, X^{\phi_n}] \\ &= \quad \mathbb{C}[\tilde{\sigma}_j] \times \mathbb{C}[X^{\phi_{k+1}}, \ldots,\, X^{\phi_n}].\end{aligned}$$

Hence the corresponding variety is also a product,

$$X(\sigma_j) = X(\tilde{\sigma}_j) \times \mathbb{C}^{n-k},$$

and this decomposition is respected by each of the gluing maps, so if $\tilde{\Delta}$ is the fan made up of the cones $\tilde{\sigma}_j$,

$$X(\Delta) \cong X(\tilde{\Delta}) \times \mathbb{C}^{n-k},$$

and this is the blow-up of $\{0\} \times \mathbb{C}^{n-k}$. □

Finally, we will demonstrate that dividing a cone up in this way using a general primitive vector $v \in \mathrm{Int}(\sigma) \cap \mathbb{Z}^n$ rather than $v_1 + \ldots + v_n$ gives us a weighted blow-up of $\mathbb{C}^n$.

This is a space constructed analogously to the blow-up, but we now quotient by a different $\mathbb{C}^*$ action. That is, the weighted blow-up of $\mathbb{C}^n$ at the origin with weights $(\alpha_1, \dots, \alpha_n)$ is

$$\begin{aligned}\hat{\mathbb{C}^n}(\alpha_1, \dots, \alpha_n) &= \{(z_1, \dots, z_n, [w_1, \dots, w_n]) \\ &\qquad \in \mathbb{C}^n \times \mathbb{P}(\alpha_1, \dots, \alpha_n) | z_i w_j^{\alpha_i} = z_j w_i^{\alpha_j} \, \forall i, j \leq n\},\end{aligned}$$

where $\mathbb{P}(\alpha_1, \dots, \alpha_n)$ is a weighted projective space.

Proposition 3.1.20. *[14] Take a simplicial non-singular cone $\sigma = \langle v_1, \dots, v_n \rangle$, and take $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ a primitive integral vector in the interior of σ. Then the fan Δ of the cones*

$$\sigma_j = \langle v_1, \dots, \hat{v_j}, \dots, v_n, v \rangle$$

has variety $X(\Delta) \cong \hat{\mathbb{C}^n}(\alpha_1, \dots, \alpha_n)$.

Proof. Again, we may assume $\sigma = \langle e_1, \dots, e_n \rangle$. Being in the interior of σ tells us that $\alpha_i > 0 \, \forall i$, and being integral that $\alpha_i \in \mathbb{Z} \, \forall i$.

As we saw in (3.1.16), each cone gives an orbifold, with coordinate ring

$$\mathbb{C}[\sigma_i] = \mathbb{C}[\{V_1^{a_1} \ldots U^d \ldots V_n^{a_n} | d + \sum \alpha_j a_j \in a_i \mathbb{N}\}].$$

We can embed these charts into the weighted blow-up as follows:

$$\begin{aligned}\Phi_i \; &: \; X(\sigma_i) \to \hat{\mathbb{C}^n}(\alpha_1, \dots, \alpha_n) \\ &\Phi_i \left(\frac{(v_1, \dots, u, \dots, v_n)}{\sim} \right) = (uv_1, \dots, u, \dots, uv_n, [v_1, \dots, 1, \dots, v_n]),\end{aligned}$$

where $X(\sigma_i) = \frac{\mathbb{C}^n}{\sim}$.

Using the transformations $X_i = U^{\alpha_i}$, $X_j^{\alpha_j} = U^{\alpha_i} V_j$, we can see that these maps are invariant under the gluings between charts and hence lift to $X(\Delta)$ to give a bijection $\Phi : X(\Delta) \to \hat{\mathbb{C}^n}(\alpha_1, \dots, \alpha_n)$. $\square$

Now we apply these constructions to problems of resolving singularities.

First we consider a toric singularity in dimension 2. We demonstrate that such a singularity can be resolved by a series of weighted blow-ups, and that this series can be deduced from the combinatorial data given by the corresponding cone. This result appears in [14] and [5].

As we have seen, a singular toric variety in 2 dimensions is an orbifold,

$$\frac{\mathbb{C}^2}{\left\langle \begin{pmatrix} \omega & 0 \\ 0 & \omega^p \end{pmatrix} \right\rangle},$$

where ω is a qth root of unity. We use results concerning continued fraction expansions of $\frac{p}{q}$ to construct a sequence of vectors which partition the cone σ into non-singular cones. Then we are able to realise this sequence as an iterated blow-up resolving the singularity in $X(\sigma)$. The number theoretic results we need are as follows, and are stated in [5] and [6]:

Proposition 3.1.21. *([5], [6]) If p, q are coprime integers, $p > 0$, and*

$$\frac{q}{p} = \cfrac{1}{e_1 - \cfrac{1}{e_2 - \cfrac{1}{\dots - \frac{1}{e_k}}}} \qquad e_i \geq 2,$$

then

$$\frac{n_{j+1}}{m_{j+1}} = \cfrac{1}{e_1 - \cfrac{1}{e_2 - \cfrac{1}{\dots - \frac{1}{e_j}}}} \qquad e_i \geq 2,$$

with m_{j+1}, n_{j+1} coprime and $m_{j+1} > 0$ defines a sequence of vectors (m_j, n_j), $j \leq k+1$ with

- $(m_0, n_0) = (0, -1)$
- $(m_{k+1}, n_{k+1}) = (p, q)$
- *and* $\det \begin{pmatrix} m_j & m_{j+1} \\ n_j & n_{j+1} \end{pmatrix} = 1 \quad 0 \leq j \leq k.$

Furthermore, we can recover the e_j from these vectors,

$$\det \begin{pmatrix} m_{j-1} & m_{j+1} \\ n_{j-1} & n_{j+1} \end{pmatrix} = e_j \qquad 1 \leq j \leq k.$$

and

$$(m_{j+1},\, n_{j+1}) = e_j(m_j,\, n_j) - (m_{j-1},\, n_{j-1})$$

Calderbank-Singer use this sequence to resolve the corresponding orbifold — the vectors give us a way to divide up the original cone into a collection of non-singular cones, and each subdivision corresponds to a blow-up operation:

Theorem 3.1.22. *[5] Let $\sigma = \langle (p,\, q),\, (0,\, -1)\rangle$. The fan, Δ, of the cones*

$$\langle (0,\, -1),\, (m_1,\, n_1)\rangle, \ldots, \langle (m_k,\, n_k),\, (p,\, q)\rangle$$

has variety $X(\Delta)$, the desingularization of $X(\sigma)$ by a series of blow-ups.

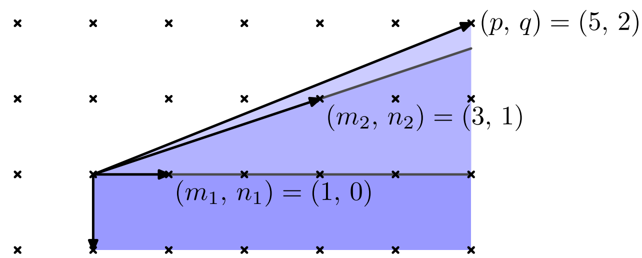


Figure 3.7: The fan Δ.

Proof. Since each cone

$$\langle (m_j,\, n_j),\, (m_{j+1},\, n_{j+1})\rangle$$

in Δ has det $\begin{pmatrix} m_j & m_{j+1} \\ n_j & n_{j+1} \end{pmatrix} = 1$, its generators form a basis of $\mathbb{Z}^2$ and hence each of the affine charts is $\mathbb{C}^2$. It remains to show that this space can be obtained by a sequence of blow-ups of $X(\sigma)$. Let

$$\Delta_j = \{\langle (0,\, -1),\, (m_1,\, n_1)\rangle, \ldots, \langle (m_j,\, n_j),\, (p,\, q)\rangle\}.$$

Each fan Δ_j is obtained by dividing the last cone in Δ_{j-1}, $\langle (m_{j-1},\, n_{j-1}),\, (p,\, q)\rangle$, into cones

$$\langle (m_{j-1},\, n_{j-1}),\, (m_j,\, n_j)\rangle \text{ and } \langle (m_j,\, n_j),\, (p,\, q)\rangle,$$

and, as we have seen, this corresponds to blowing up a point in $X(\Delta_k)$. $\square$

We can similarly resolve more complex singularities in higher dimensions:

Theorem 3.1.23. *[14] If*

$$\sigma = \langle e_2, \ldots, e_n, \alpha_1 e_1 - \alpha_2 e_2 - \ldots - \alpha_n e_n \rangle \subseteq \mathbb{R}^n$$

with $\mathrm{hcf}(\alpha_1, \ldots, \alpha_n) = 1$ *and* $\alpha_1 > \alpha_i \geq 0 \ \forall 1 < i \leq n$ *then*

$$X(\sigma) \cong \frac{\mathbb{C}^n}{G},$$

where G *is the group generated by*

$$\begin{pmatrix} \omega & & & \\ & \omega^{\alpha_2} & & \\ & & \ddots & \\ & & & \omega^{\alpha_n} \end{pmatrix},$$

and ω *is an* α_1*th root of unity and the orbifold singularity can be resolved by a series of weighted blow-ups.*

Proof. This result appears in [14], although the approach we use here to prove it is different. The first statement is (3.1.16). Note that in particular, $X(\sigma)$ is an orbifold of degree α_1. We can reduce the degree of the singularity as follows:

Form the fan Δ of

$$\begin{aligned} \sigma_1 &= \langle e_1, \ldots e_n \rangle \\ \sigma_i &= \langle e_1, \ldots, \hat{e}_i, \ldots, e_n, \alpha_1 e_1 - \alpha_2 e_2 - \ldots - \alpha_n e_n \rangle \quad 2 \leq i \leq n \end{aligned}$$

so that $X(\Delta)$ is a weighted blow-up of $X(\sigma)$. The resulting charts are $X(\sigma_1)$, which is non-singular, and $X(\sigma_i)$ for $2 \leq i \leq n$, which is an orbifold of degree α_i. In particular each of these degrees is smaller than α_1.

Now fix $2 \leq i \leq n$ and suppose that σ_i is not non-singular. We show that after a suitable change of basis we can express the cone σ_i in the same form as σ.

Let

$$\begin{aligned} \alpha_1 &= \mu_1 \alpha_i - r_1, \qquad \mu_1, r_1 \in \mathbb{Z}, \text{ and } 0 \leq r_1 < \alpha_i \\ \alpha_j &= -\mu_j \alpha_i + r_j \qquad j \geq 2, \mu_j, r_j \in \mathbb{Z} \text{ and } 0 \leq r_j < \alpha_i. \end{aligned}$$

Then consider the new basis

$$f_1 = e_1, \ldots, f_{n-1} = e_{n-1}, f_n = -e_n + \mu_1 e_1 + \ldots + \mu_{n-1} e_{n-1}.$$

Then

$$\sigma_i = \langle f_1, \ldots, \hat{f}_i, \ldots, f_{n-1}, \alpha_i f_i - r_1 f_1 - r_2 f_2 - \ldots - r_n f_n \rangle.$$

Then we can apply this process inductively until we are left with only non-singular cones. If the fan of these cones is $\tilde{\Delta}$ then $X(\tilde{\Delta})$ is a resolution of $X(\sigma)$ by a sequence of weighted blow-ups. □

3.2 Polytopes and symplectic toric manifolds

Now consider a manifold M^{2n} equipped with a symplectic form ω and a Hamiltonian T^n action, that is, ω is a non-degenerate closed 2-form and $G = T^n$ acts on M in such a way that there exists an equivariant map $\mu : M \to \mathfrak{g}^*$ which has

$$-d\langle \mu, v \rangle = \omega(v, \cdot) \quad \forall v \in \mathfrak{g}$$

where $\mathfrak{g}$ is the Lie algebra of G, $\mathfrak{G}^*$ its dual, and the pairing is the natural pairing of $\mathfrak{g}^*$ with $\mathfrak{g}$. We say μ is a *moment map* of the group action, and M is a *toric symplectic manifold.*

Example 3.2.1. • $\mathbb{C}^n$, *with the standard symplectic form*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

and T^n acting by rotations. In polar coordinates the symplectic form becomes

$$\omega = \sum_{i=1}^{n} r_i dr_i \wedge d\theta_i,$$

so the ith component of the moment map satisfies

$$\begin{aligned} d\mu_i &= -\left(\sum_{j=1}^{n} dr_j \wedge \theta_j\right)\left(\frac{\partial}{\partial \theta_i}, \cdot\right) \\ &= r_i dr_i. \end{aligned}$$

Then

$$\mu = \frac{1}{2}\sum_{i=1}^{n} r_i^2 d\theta_i = \frac{1}{2}\sum_{j=1}^{n} |z_i|^2 d\theta_i$$

is a moment map for this action, and the image of this moment map

$$\mu(\mathbb{C}^n) = \{(m_1, \ldots, m_n) | m_1, \ldots, m_n \geq 0\}.$$

- $\mathbb{C}P^n$, *with torus action*

$$(\theta_1, \ldots, \theta_n) \cdot [w_0, \ldots, w_n] = [w_0, e^{i\theta_1} w_1, \ldots, e^{i\theta_n} w_n]$$

and the standard symplectic form ([18], cf. (2.2.4)) on the chart $w_0 = 1$,

$$\omega = \frac{i}{2\pi}\left(\frac{\sum_{j=1}^n dw_j \wedge d\overline{w_j}}{1 + \sum_{j=1}^n |w_j|^2} - \frac{(\sum_{j=1}^n \overline{w_j} dw_j) \wedge (\sum_{k=1}^n w_k d\overline{w_k})}{(1 + \sum_{j=1}^n |w_j|^2)^2}\right).$$

Note that in polar coordinates

$$\begin{aligned} dw_j &= e^{i\theta_j} dr_j + i\theta_j r_j e^{i\theta_j} d\theta_j \\ d\overline{w_j} &= e^{-i\theta_j} dr_j - i\theta_j r_j e^{-i\theta_j} d\theta_j \end{aligned}$$

and contracting in $\frac{\partial}{\partial\theta_j}$ *now gives*

$$\begin{aligned} d\mu_j &= \frac{1}{2\pi}\left(2ir_j \frac{\Re(e^{i\theta_j} d\overline{w_j})}{1 + \sum_{k=1}^n r_k^2} - 2ir_j^2 \frac{\Re(\sum_{k=1}^n w_k d\overline{w_k})}{(1 + \sum_{k=1}^n r_k^2)^2}\right) \\ &= \frac{1}{2\pi}\left(\frac{w_j d\overline{w_j} + \overline{w_j} dw_j}{1 + \sum_{k=1}^n |w_k|^2} - \frac{|w_j|^2}{(1 + \sum_{k=1}^n |w_k|^2)^2} \sum_{k=1}^{n} (w_k d\overline{w_k} + \overline{w_k} dw_k)\right) \end{aligned}$$

so

$$\mu = \frac{1}{1 + \sum_{k=1}^n |w_k|^2} (|w_1|^2, \ldots, |w_n|^2)$$

is a moment map for this action.

The image of this chart under the moment map is then

$$\{(m_1, \ldots, m_n) | m_1, \ldots, m_n \geq 0, m_1 + \ldots + m_n < 1\}.$$

Considering the other charts gives the remaining points, so that

$$\mu(M) = \{(m_1, \ldots, m_n) | m_1, \ldots, m_n \geq 0, m_1 + \ldots + m_n \leq 1\},$$

the standard simplex in $\mathbb{R}^n$.

We show that we can construct toric symplectic manifolds from convex polytopes in $\mathbb{R}^n$, using the presentation of the Delzant theorem in [20]. We discuss the Atiyah-Guillemin-Sternberg convexity theorem ([1], [21], further described in [20]), which tells us that every compact symplectic toric manifold is given in this way.

In order to associate a toric symplectic manifold to a compact Delzant polytope, we consider symplectic reductions of $\mathbb{C}^d$. Given a compact Delzant polytope, P, it is possible to find such a reduction such that the image of the moment map is P. The role of the Delzant condition here is to ensure that the symplectic reduction produces a smooth manifold.

Definition 3.2.2. *Let* $\Delta \subseteq \mathbb{R}^n$ *be a rational convex polytope,*

$$\Delta = \cap_{i \in I} \{x \in \mathbb{R}^n | u_i \cdot x = -\lambda_i\} = \cap_{i \in I} U_i \qquad u_i \in \mathbb{Z}^n,\ \lambda_i \in \mathbb{Z}.$$

Then Δ *is Delzant if for every subset* $J \subseteq I$ *for which the bounding hyperplanes intersect,*

$$\cap_{i \in J} \{x \in \mathbb{R}^n | u_i \cdot x \leq -\lambda_i\} \neq \emptyset,$$

the normals $\{u_i | i \in J\}$ *can be extended to a* $\mathbb{Z}$*-basis on* $\mathbb{Z}^n$.

It is then sufficient to check this condition at the vertices of a polytope, since this implies the condition at higher dimensional faces.

We now show that every Delzant polytope is the image of a toric symplectic manifold under a moment map. In order to show this we use the polytope to find a symplectic reduction of $\mathbb{C}^d$ with a Hamiltonian torus action, and show that the image of this manifold under a moment map is the original polytope. The Delzant condition is required to show this symplectic reduction is a manifold. This is the Delzant theorem [9], and we use the exposition of this result in [20].

Theorem 3.2.3. *[9] Given a compact Delzant polytope*

$$\Delta = \cap_{i=1}^{d} \{x \in \mathbb{R}^n | u_i \cdot x + \lambda_i \geq 0\},$$

there is a compact symplectic toric manifold M^{2n} with moment map Ψ such that

$$\Delta = \Psi(M).$$

Proof. Define a map

$$\pi_* : \mathbb{R}^d \to \mathbb{R}^n \qquad \pi_*(e_i) = u_i.$$

Since the polytope is compact this map is surjective. Letting $\mathfrak{g} = \ker \pi_*$ gives us a short exact sequence

$$0 \to \mathfrak{g} \to \mathbb{R}^d \xrightarrow{\pi_*} \mathbb{R}^n \to 0$$

and dual to this, another sequence

$$0 \to \mathbb{R}^n \xrightarrow{\pi^*} \mathbb{R}^d \xrightarrow{\rho} \mathfrak{g}^* \to 0$$

where ρ is the restriction map and

$$\pi^*(x) = (u_1 \cdot x, \ldots, u_d \cdot x).$$

Consider the first sequence — this generates a sequence of Lie groups,

$$0 \to G \to T^d \xrightarrow{\pi} T^n \to 0.$$

T^d acts on $\mathbb{C}^d$, and this action has moment map

$$\Phi : \mathbb{C}^d \to \mathbb{R}^d \qquad \Phi(z_1, \ldots, z_d) = (|z_1|^2 - \lambda_1, \ldots, |z_d|^2 - \lambda_d).$$

Restricting to G we get an action with moment map

$$\Phi_G = \rho \circ \Phi : \mathbb{C}^d \to \mathfrak{g}^*$$

Then define

$$M = \frac{\Phi_G^{-1}(0)}{G},$$

the symplectic reduction of $\mathbb{C}^d$ by G. Suppose M is a manifold (we will show later that this follows from the Delzant condition). T^n acts on this space with moment map

$$\Psi = (\pi^*)^{-1} \circ \Phi : M \to \mathbb{R}^n.$$

Note that π^* is invertible on $\Phi(M)$ because π^* is injective and $\operatorname{Im}\Phi \subseteq \ker\rho = \operatorname{Im}\pi^*$. But

$$\Phi_G^{-1}(0) = \Phi^{-1}(\ker\rho) = \Phi^{-1}(\operatorname{Im}\pi^*),$$

so the image of this space under Ψ is

$$\begin{aligned}
\Psi(M) &= (\pi^*)^{-1}(\operatorname{Im}\Phi) \\
&= (\pi^*)^{-1}(\{(y_1, \ldots, y_d) \in \mathbb{R}^d | y_1 \geq -\lambda_1, \ldots, y_d \geq -\lambda_d\}) \\
&= \{x \in \mathbb{R}^n | u_i \cdot x \geq -\lambda_i \quad \forall i \leq d\} = \Delta.
\end{aligned}$$

It just remains to show that M is indeed a manifold. We do this by showing that the stabiliser of any point in $\Phi_G^{-1}(0)$ intersects trivially with G. Take $z \in \Phi_G^{-1}(0)$ and let

$$I = \{i \leq d | z_i = 0\}.$$

Then the T^d-stabiliser of z is

$$T^I = \{(\theta_1, \ldots, \theta_d) | \theta_i = 0 \quad \forall i \notin I\}.$$

since $\Phi_G(z) = 0$, $\Phi(z) \in \ker\rho = \operatorname{Im}\pi_*$, so for some $x \in \mathbb{R}^n$

$$\Phi(z) = \pi_*(x).$$

Then

$$u_i \cdot x = \lambda_i \iff z_i = 0,$$

so x lies on exactly those faces $\{y | u_i \cdot y = \lambda_i\}$ for which $i \in I$. By the Delzant condition we can then extend $\{u_i\}_{i \in I}$ to a $\mathbb{Z}$-basis. Let $f_1, \ldots, f_n$ be such a basis, with $\{f_1, \ldots, f_k\} = \{u_i\}_{i \in I}$, so that

$$T^I = \{(\theta_1, \ldots, \theta_k, 0, \ldots, 0) | \theta_i \in S^1\}.$$

Then in this basis

$$\begin{aligned}\pi|_{T^I} \quad : \quad & T^I \to T^n \\ & \pi|_{T^I}(\theta_1, \ldots, \theta_k, 0, \ldots, 0) = [\theta_1 f_1 + \ldots + \theta_k f_k]\end{aligned}$$

is injective. Hence

$$\mathrm{Stab}_G(z) = \mathrm{Stab}_{\mathbb{C}^d}(z) \cap \ker \pi = \{0\}$$

and the action of G is locally free. Since G is compact the action is proper, therefore the quotient M is a manifold. □

We saw during this proof that the image of these symplectic toric manifolds under a moment map is a convex polytope. We now turn to the Atiyah-Guillemin-Sternberg convexity theorem, which will show that this is always the case. This result was proved independently by Atiyah [1] and Guillemin-Sternberg [21], but here we will present the approach of Atiyah. In particular, this result shows compact polytopes classify the compact symplectic toric manifolds.

This is proved in two stages, showing first that for a Hamiltonian torus action of any dimension the fibres of the moment map are connected. This proof relies heavily on Morse theory and here we will only sketch it. We then use this fact to show the image of the moment map is convex.

Lemma 3.2.4. *[1] If $\mu : M \to \mathbb{R}^n$ is a moment map of a Hamiltonian torus action, $\forall c \in \mathbb{R}^n$, $\mu^{-1}(c)$ is connected.*

Proof. This is shown by considering each of the components of μ in turn. Each is a Morse-Bott function with critical manifold only of even index, and hence has connected fibre. When we add a new component, this still has the same property, and it follows inductively that that $\mu^{-1}(x_1, \ldots, x_n) = \mu_1^{-1}(x_1) \cap \ldots \cap \mu_n^{-1}(x_n)$ is also connected. □

Proposition 3.2.5. *[1] Let $\mu : M \to \mathbb{R}^n$ be a moment map of a Hamiltonian torus action. The image of the moment map is convex.*

Proof. Take any line $\ell \subseteq \mathbb{R}^n$. There is a projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and a point $c \in \mathbb{R}^{n-1}$ such that $\ell = \pi^{-1}(c)$. Now define

$$\nu : M \to \mathbb{R}^{n-1} \qquad\qquad \nu = \pi \circ \mu.$$

This is the moment map of some T^{n-1} action on M, so by the previous lemma $\nu^{-1}(c)$ is connected. Then

$$\mu(M) \cap \ell = \mu(M) \cap \pi^{-1}(c) = \mu(\nu^{-1}(c))$$

is connected. Since its intersection with any line is connected, $\mu(M)$ is convex. $\square$

In fact Atiyah describes $\mu(M)$ more explicitly:

Proposition 3.2.6. *[1] Let $\mu : M \to \mathbb{R}^n$ be a moment map of a Hamiltonian torus action. The image of the moment map is the convex hull of the fixed points of the torus action.*

Proof. Let Z be the set of critical points of μ. Since the components of μ are Morse-Bott functions this is a disjoint union of submanifolds, and since $d\mu|_Z = 0$, μ is constant on each of the components, hence $\mu(Z)$ is a finite collection of points.

Now consider a linear combination of the components of μ,

$$\phi = \sum_{i=1}^{n} \lambda_i \mu_i.$$

For almost every choice of $(\lambda_1, \dots, \lambda_n)$, the critical set of ϕ lies inside Z and in particular ϕ attains its maximum here. If $\mu(M)$ contained a point x outside the convex hull of $\mu(Z)$ it would be possible to find $(\lambda_1, \dots, \lambda_n)$ with a neighbourhood $U \subseteq \mathbb{R}^n$ for which the functionals

$$\tilde{\phi} = \sum_{i=1}^{n} \tilde{\lambda}_j \mu_j \qquad (\tilde{\lambda}_1, \dots, \tilde{\lambda}_n) \in U$$

attain their maxima near x. By contradiction no such x exists. Since $\mu(M)$ is convex, it must, then, be the convex hull of $\mu(Z)$. $\square$

It remains to find the stabilisers. Take a point $x \in M$ and suppose $\pi(x)$ lies on k faces (in two dimensions, $k = 0$, 1, or 2). Up to a change of basis we can assume $\mu(M)$ locally has the form

$$\mu(M) = \{(0, \dots, 0, y_{k+1}, \dots, y_n) | y_{k+1}, \dots, y_n > 0\}.$$

Now consider a neighbourhood $U \subseteq M$ of x. Since $\pi(x)$ is a minimum of $\mu_1, \dots, \mu_k$, and using the definition of the moment map,

$$0 = -d\mu_i(x) = \omega(e_i, \cdot) \quad \forall i \leq k.$$

Since ω is non-degenerate it follows that $e_i = 0$ and the circle generated by $\frac{\partial}{\partial \theta_i}$ fixes x. For the same reason it is clear that the circles generated by $\frac{\partial}{\partial \theta_j}$, $j > k$ do not fix x.

In particular, in two dimensions this tells us the vertices are fixed points and the edges are stabilised by

$$\{(\theta_1, \theta_2) \in T^2 | \underline{n} \cdot (\theta_1, \theta_2) = 0\}.$$

Then the image of any symplectic toric manifold under a moment map is a polytope, and from this polytope we can construct a symplectic toric manifold symplectomorphic to the original by reduction of $\mathbb{C}^d$. In particular, compact Delzant polytopes correspond with compact symplectic toric manifolds.

3.3 Topology of toric 4-manifolds

We have seen how we can construct algebraic and symplectic toric spaces from combinatorial data, and now ask how these constructions are related. We restrict our attention to 4-manifolds, and show how the topology of a toric manifold can be classified by a result of Orlik-Raymond [27]. We then use this classification to show how, given a symplectic toric 4-manifold, we can find a fan yielding a homeomorphic toric variety. Finally we give a combinatorial argument of Fulton [14] which shows that, in dimension two, the polytopes correspond with the fans seen in the first section (3.1).

So we turn to the topological classification result, which is due to Orlik-Raymond [27] and relies on the differentiable slice theorem, which can be found in [3]. First consider a point x in a smooth 4-manifold with a smooth effective action of T^2. We can classify the possible stabilisers of this point, as well as the behaviour of the action near this point as follows:

Proposition 3.3.1. *[27] Take a smooth, effective action of T^2 on a 4-manifold. The stabiliser of every point $x \in M$ is one of the following:*

1. $\{e\}$

2. $\mathbb{Z}_n \times \{e\}$

3. $\mathbb{Z}_n \times \mathbb{Z}_m$

4. $G(m, n)$

5. *or* T^2,

where we introduce the notation

$$G(m, n) = \{(\theta_1, \theta_2) | m\theta_1 + n\theta_2 = 0\}$$

to specify 1*-dimensional subgroups of the torus. Furthermore, the image of such points in the orbit space* M/T^2 *must be, respectively,*

1. *an interior point*

2. *an isolated interior point*

3. *an isolated interior point*

4. *a boundary point lying on a curve of points with the same stabiliser*

5. *an isolated boundary point.*

Proof. Let H be the stabiliser of x. From the slice theorem [3], we can find a neighbourhood U of $e \in T^2/H$ and a map $\chi : U \to T^2$ such that $\pi \circ \chi = \mathrm{id}$, where π is the quotient map, and a subset $S \subseteq M$ invariant under H such that

$$g \cdot S \cap S = \emptyset$$

for all $T^2 - H$, and a map

$$F : U \times S \to M, \qquad F([g], y) = \chi([g]) \cdot y$$

which is a homeomorphism onto its image. Furthermore, S is homeomorphic to a disc $D^{2+\dim(H)}$, on which H acts as a group of orthogonal linear transformation.

Now suppose we have a point $x \in M$ with stabiliser $\mathbb{Z}_k \times G(m, n)$. Then we would have a linear action of $\mathbb{Z}_k \times G(m, n)$ on a slice D^3. Since $SO(3)$ has no such subgroup, no such point x can exist and the listed stabilisers are the only possibilities.

Then consider the models this gives of the space near x.

If $H = \{e\}$, $\mathbb{Z}_n \times \{e\}$ or $\mathbb{Z}_n \times \mathbb{Z}_m$ the slice is D^2 so such a point must lie in the interior of the orbit space. In the second and third cases this disc is acted on by rotations, of which the origin is the only fixed point, so these orbits must be isolated.

If $H = T^2$ the slice is D^4 and the torus acts by rotations about two axes, so x again lies on the boundary, and the origin is the only fixed point of the H-action so these orbits must be isolated.

Similarly if $H = G(m, n)$, the slice is D^3 acted on by rotation and must lie on the boundary of the orbit space, and lies on a curve in the orbit space of points stabilised by $G(m, n)$. □

Then the orbit space can be described by a diagram consisting of a 2-manifold with boundary, whose boundary is partitioned into a finite number of arcs and fixed points, each arc labelled by a primitive vector $\pm(m, n)$ representing its stabiliser $G(m, n)$, and possibly a collection of interior points labelled with finite stabilisers $\mathbb{Z}_n \times \mathbb{Z}_m$.

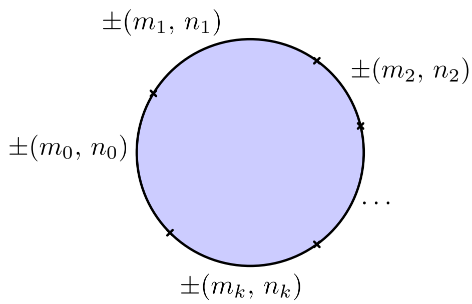
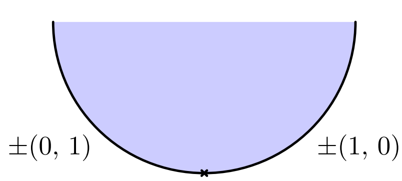


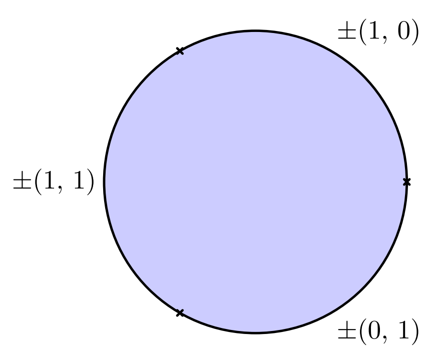
Figure 3.8: Orlik-Raymond diagram.

Example 3.3.2.

- *For a symplectic toric manifold we can read the Orlik-Raymond diagram off from the image of the moment map,*



(a) Orbit space of $\mathbb{C}^2$.

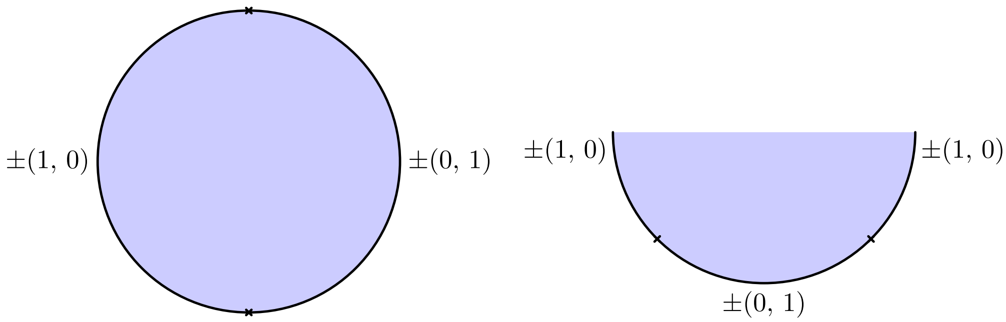


(b) Orbit space of $\mathbb{C}P^2$.

- S^4. *If we view S^4 as the one point compactification of $\mathbb{C}^2$, the torus action is that of $\mathbb{C}^2$ extended by fixing the extra point, so we obtain diagram 3.9(c).*

- $S^2 \times \mathcal{H}^2$. *The points with non-trivial stabiliser in this space are*
  - $S^2 \times \{0\}$, *fixed by* $G(0,\,1)$ *and*
  - $\{(0,\,0,\,1)\} \times \mathcal{H}^2$ *and* $\{(0,\,0,\,-1)\} \times \mathcal{H}^2$, *fixed by* $G(1,\,0)$.

  Then the orbit space is shown in diagram 3.9(d).



(c) Orbit space of S^4. (d) Orbit space of $S^2 \times \mathcal{H}^2$.

We next show that such a diagram classifies 4-manifolds with a smooth T^2 action, subject to two constraints — that there are no finite stabilisers and that any submanifold on which T^2 acts freely is a trivial T^2 bundle. We do this by showing that we can construct a cross-section to the quotient map, that is, a continuous map $\chi : X^* \to X$ with $\pi \circ \chi = \mathrm{id}$. We do this by first proving we can find cross-sections on two model spaces, then decompose the full space into such model spaces and patching the cross-sections together. Such a map will allow us to identify a 2-manifold in each space transverse to the torus action, then use the action to generate a homeomorphism between them.

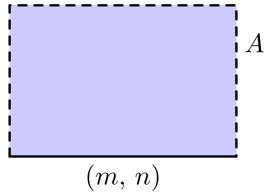
Lemma 3.3.3. *[27] Let X be a 4-manifold with smooth torus action and orbit space X^*. We can extend a cross-section $\tilde{\chi}$ on a subset $A \subseteq X^*$ to X^* in the following pairs $(X^*,\,A)$:*

1. $X^* = I \times I$, *with stabiliser groups*

$$\operatorname{Stab} x = \begin{cases} G(m,\,n) & \pi(x) = (\lambda,\,0) \\ \{e\} & \textit{otherwise} \end{cases}$$

   and

$$A = (\{0\} \times I) \cup (\{1\} \times I) \cup (I \times \{1\}).$$



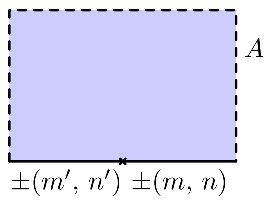
2. $X^* = I \times I$

$$\operatorname{Stab} x = \begin{cases} G(m,\, n) & \pi(x) = (\lambda,\, 0),\ \lambda > \frac{1}{2} \\ G(m',\, n') & \pi(x) = (\lambda,\, 0),\ \lambda < \frac{1}{2} \\ T^2 & \pi(x) = (0,\, 0) \\ \{e\} & \textit{otherwise} \end{cases}$$

and

$$A = (\{0\} \times I) \cup (\{1\} \times I) \cup (I \times \{1\}).$$

for some primitive vectors $(m,\, n)$ *and* $(m',\, n')$, *with* $\det \begin{pmatrix} m & m' \\ n & n' \end{pmatrix} = \pm 1.$



Proof. 1. Let $(m',\, n')$ be a primitive vector with $\det \begin{pmatrix} m & m' \\ n & n' \end{pmatrix} = \pm 1$. Then $G(m',\, n')$ acts freely on X, and we can form the quotient, $\bar{\pi} : X \to Y$. The action of $G(m,\, n)$ and the cross-section $\tilde{\chi}$ then descends to this space, $\bar{\chi} = \bar{\pi} \circ \tilde{\chi} : A \to Y$. Because this space is contractible there is no obstruction to extending this map to $X \setminus \pi^{-1}(I \times \{0\})$ and we can find an extension $\chi : X^* \to X$.

2. Let $X_1^* = [0,\, \frac{1}{2}] \times I$ and $X_2^* = [\frac{1}{2},\, 1] \times I$. We can extend the cross-section $\tilde{\chi}|_{A \cap X_1^*}$ to $A_1 = (\{0\} \times I) \cup (\{\frac{1}{2}\} \times I) \cup ([0,\, \frac{1}{2}] \times \{1\})$, and by part (1), this extends to a cross-section $\chi_1 : X_1^* \to \pi^{-1}(X_1^*)$. Similarly, we can extend the values of χ_1 on $\{\frac{1}{2}\} \times I$ and $\tilde{\chi}$ on $A_2 = (\{1\} \times I) \cup ([\frac{1}{2},\, 1] \times \{1\})$ to a cross-section χ_2 on X_2^* in such a way that

$$\chi : X^* \to X \qquad \chi(x) = \begin{cases} \chi_1(x) & x \in X_1^* \\ \chi_2(x) & x \in X_2^* \end{cases}$$

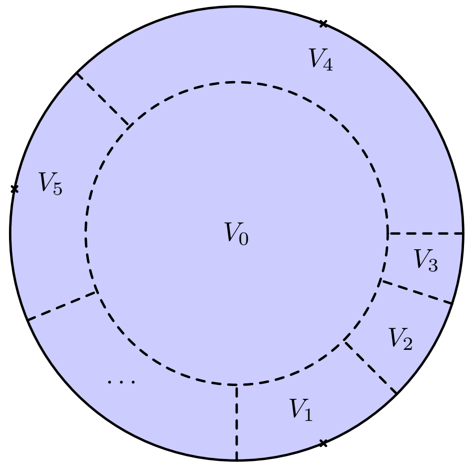


Figure 3.9: Dividing the Orlik-Raymond diagram.

is well-defined, and this is then the required cross-section.

$\square$

Now we can use these results to piece together a cross-section of a T^2 action on a 4-manifold:

Proposition 3.3.4. *[27] Let T^2 act smoothly on a compact 4-manifold M, in such a way that*

- *no point has a finite stabiliser and*
- *if $U \subseteq M$ is a subset of M on which T^2 acts freely, then the fibre bundle $M \to \frac{M}{T^2}$ is trivial.*

Then there exists a cross-section of the T^2 action on M.

Proof. We have seen that the orbit space M^* is a 2-manifold with boundary, that the boundary consists of a finite number of fixed points connected with arcs of points with stabiliser $G(m, n)$.

Take a neighbourhood, V, of the boundary and divide it up into a finite collection of compact sets V_i with disjoint interiors, each homeomorphic to $\overline{D^2}$, such that each V_i contains at most one fixed point and let $V_0 = M^* \setminus \bar{V}$, as in diagram 3.9. By (ii), $\pi^{-1}(V_0)$ is a trivial bundle, so we can find a cross-section here, $\chi|_{\bar{V_0}}$. Now take V_1 and extend $\chi|_{\bar{V_0} \cap \partial \bar{V_1}}$ to $\partial V_1 \setminus \partial M^*$, and then by the lemma we can extend χ to V_1. Then extend this cross-section to V_2 in the same way. Proceeding in this way we can extend χ to each of the V_i until we have the desired cross-section on all of M. $\square$

Finally we use these cross-sections to show that the combinatorial data encapsulated in the diagrams described in the discussion following (3.3.1) classifies such actions.

Theorem 3.3.5. *If M_1, M_2 are two 4-manifolds with smooth effective T^2 actions such that*

- *no point has a finite stabiliser and*
- *if $U \subseteq M$ is a subset of M on which T^2 acts freely, then the fibre bundle $M \to \frac{M}{T^2}$ is trivial.*

and there is a homeomorphism $\Phi : M_1^* \to M_2^*$ and an element $A \in SL(2, \mathbb{Z})$ such that if $x \in \partial M_1^*$, has stabiliser $G(m, n)$ then $\Phi(x)$ has stabiliser $G((m, n)A^T)$, then Φ lifts to an equivariant homeomorphism

$$\Psi : M_1 \to M_2.$$

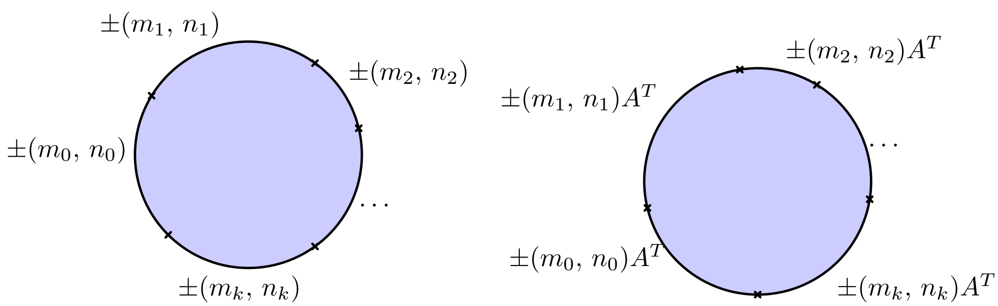


Figure 3.10: Two equivalent Orlik-Raymond diagrams.

In particular, the orbit diagrams (Figure 3.8) up to homeomorphism and change of basis of $\mathbb{Z}^2$ classify such spaces up to equivariant homeomorphism.

Proof. Take cross-sections $\chi_1 : M_1^* \to M_1$ and $\chi_2 : M_2^* \to M_2$. Define

$$\begin{aligned} \Psi \quad : \quad & M_1 \to M_2, \\ & \Psi(x) = \chi_2 \circ \Phi \circ \pi_1(x) \qquad\qquad \forall x \in \operatorname{Im} \chi_1 \end{aligned}$$

$$\begin{tikzcd} M_1 \arrow[r, "\Psi"] \arrow[d, "\pi_1"] & M_2 \\ M_1^* \arrow[r, "\Phi"] & M_2^* \arrow[u, "\chi_2"'] \end{tikzcd}$$

and extend this by

$$\Psi((m,\, n) \cdot x) = ((m,\, n)A^T) \cdot \Psi(x).$$

This map is well-defined because Φ identifies the two orbit structures together, and is the required equivariant homeomorphism. $\square$

We have seen how we can build toric varieties from fans and toric symplectic manifolds from polytopes. We will now use the previous result classifying the topological structures of 4-manifolds with smooth effective T^2 actions to relate these two constructions — we show how we may take a two dimensional polytope P and build a fan whose variety is homeomorphic to the toric symplectic manifold of the polytope, X_P.

This is in fact a special case of a result (given in [14]) which holds in all dimensions — given any Delzant polytope P, the collection of cones over the faces of the dual polytope gives a fan whose toric variety is homeomorphic (in fact analytically isomorphic) to the symplectic toric manifold. This general result is shown by using a collection of sections to embed the toric variety into a projective space. It can then be shown that the image is a symplectic submanifold, and the image of the restriction of the standard moment map recovers the original polytope.

Let $P \subseteq \mathbb{R}^2$ be a Delzant polytope and denote the normals to its faces $\{\underline{w}_j\}_{j=1}^d$. We denote $\underline{w}_{d+1} = \underline{w}_1$ to simplify notation. Let P^0 be the convex hull of this set, and Δ the fan consisting of the cones over its faces. Then the two dimensional cones in Δ are

$$\sigma_j = \langle \underline{w}_j,\, \underline{w}_{j+1} \rangle \quad 0 \leq j \leq d.$$

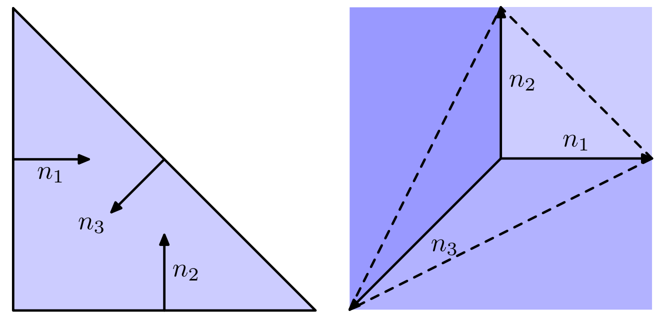
Example 3.3.6. *We have seen (3.2.1) that the polytope of $\mathbb{C}P^2$ is the standard simplex in $\mathbb{R}^2$. So as figure 3.11 shows, applying this construction recovers the fan of (3.1.14).*

By the Delzant condition each such pair forms a basis of $\mathbb{Z}^2$, so the corresponding chart $\mathbb{C}[\sigma_j]$ is $\mathbb{C}^2$. To find the stabilisers under torus actions, note that

$$\check{\sigma} = \langle (a,\, b),\, (c,\, d) \rangle$$

where $(a,\, b) \cdot \underline{w}_j = 0$ and $(c,\, d) \cdot \underline{w}_{j+1} = 0$. Then

$$\mathbb{C}[\sigma_j] = \mathbb{C}[X_j,\, Y_j]$$



(a) The polytope of $\mathbb{C}P^2$ and its normals.
(b) The fan generated by the normals.

Figure 3.11: Obtaining the fan of $\mathbb{C}P^2$ from its polytope.

with

$$\begin{aligned}(\theta_1, \theta_2) \cdot X_j &= e^{i(a\theta_1 + b\theta_2)} X_j \\ (\theta_1, \theta_2) \cdot Y_j &= e^{i(c\theta_1 + d\theta_2)} Y_j.\end{aligned}$$

Then $X(\sigma_j) \cong \mathbb{C}^2$ with coordinates (x_j, y_j) and the axis $x_j = 0$ has stabiliser $G(\underline{w}_j)$, and the axis $y_j = 0$ has stabiliser $G(\underline{w}_{j+1})$. Now consider $X(\sigma_{j+1})$ — this is another $\mathbb{C}^2$ with axes stabilised by $G(\underline{w}_{j+1})$ and $G(\underline{w}_{j+2})$. We will show that the gluing map between the two charts identifies the two axes with stabiliser $G(\underline{w}_{j+1})$.

The key fact here is that $X_{j+1} = Y_j^{-1}$. Then the gluing map is

$$\begin{aligned}\Phi \ &: \ X(\sigma_j) \setminus \{y_j = 0\} \to X(\sigma_{j+1}) \setminus \{x_{j+1} = 0\}, \\ &\quad \Phi(x_j, y_j) = \left(\frac{1}{y_j}, x_j^\alpha y_j^\beta\right)\end{aligned}$$

for some $\alpha, \beta \in \mathbb{Z}$, $\alpha > 0$. In particular the axis $\{y_j = 0\} \subseteq X(\Delta_j)$ is taken to $\{x_{j+1} = 0\} \subseteq X(\Delta_{j+1})$, as claimed.

Now consider the orbit space of $X(\Delta)$. Starting with one chart, $X(\sigma_1)$, we can build up a picture of this — we begin with a quadrant with the two axes stabilised by $G(\underline{w}_1)$ and $G(\underline{w}_2)$. When we glue in $X(\sigma_2)$ we add a new component to this boundary with stabiliser $G(\underline{w}_3)$, and so on. When we glue in the final chart the boundary forms a closed loop, making the orbit space $X(\Delta)^*$ a disc with boundary divided into d segments, labelled with stabilisers $G(\underline{w}_1), G(\underline{w}_2), \dots, G(\underline{w}_d)$.

In particular this is the same as the orbit space of X_P, the toric symplectic manifold of P. Then by (3.3.5) the two spaces are homeomorphic.

Chapter 4

Self-dual toric metrics

Having considered the topology of 4-dimensional toric manifolds, we review a construction of Joyce [23] which allows us to explicitly construct a conformal class of self-dual metrics on 4-manifolds with an effective T^2 action.

Joyce does this by first constructing a conformal class of self-dual metrics on a product space $D^2 \times T^2$, before showing that with certain explicit choices of data in this construction the condition for self-duality reduces to a system of linear PDEs. Subject to certain asymptotic conditions on the choice of conformal factor, the metric built on the union of the regular orbits of the torus action extends to the degenerate fibres and gives us a metric on the toric 4-manifold. Joyce then finds a family of solutions, linear combinations of which allow us to reconstruct the combinatorial diagrams of the previous chapter, following the approach of [5].

4.1 The Joyce equations

We begin by quoting without proof a condition ((2.4.2), [23]) for a metric on $M = N \times T^2$ to be self-dual, where N is a contractible surface:

Theorem 4.1.1. *[23] Let h be a metric on a surface N of scalar curvature -1, and ∇^N the Levi-Civita connection. Regarding TN as a complex line bundle with complex structure J, let L be a complex line bundle with an identification*

$$L \otimes_{\mathbb{C}} L \cong TN.$$

This identification corresponds to a real identification

$$S_0^2 L \cong TN,$$

where S_0^2 *is the part of the symmetric square of* L *with determinant* 0*, and there is a natural section* C *of* $S_0^2 L \otimes T^*N$ *given by the identity section under the identification*

$$S_0^2 \otimes TN \cong T^*N \otimes TN.$$

We normalise this so that $\|C\|^2 = \frac{1}{4}$ *in the induced metric on* $L^* \otimes TN \otimes L^*$*. Let* $\tilde{h}$*,* ∇^L *and* $\tilde{J}$ *be the metric, connection and complex structure induced on* L *by this identification.*

If $\phi \in \Gamma(L^* \otimes \mathbb{R}^2)$ *is a non-degenerate and orientation preserving section such that*

$$\nabla^L_a \phi_\beta + J_a^c \tilde{J}_\beta^\delta \nabla^L_c \phi_\delta = 2\phi_\gamma h^{\gamma\,\delta} h_{a\,e} C^e_{\delta\,\beta}, \tag{4.1}$$

where the latin indices run over a basis of TN *or* T^*N*, as appropriate, while the greek indices run over a basis of* L *or* L^**. Then*

$$[g] = [h + \tilde{h}],$$

where we identify L *with* $T(T^2)$ *via* ϕ*, is a conformal class of self-dual metrics on* $N \times T^2$*.*

This result is proved by considering the curvature and torsion of connections which preserve a conformal metric.

In particular, if we identify N with the upper half plane with the hyperbolic metric,

$$\mathcal{H}^2 = \{(\rho, \eta) | \rho > 0\} \qquad\qquad g = \frac{d\rho^2 + d\eta^2}{\rho^2},$$

we fix the complex structure and are left with only ϕ to choose. Considering this special case allows us to express this condition explicitly, and to find explicit solutions to it.

Theorem 4.1.2. *[23] Let* $\phi_1, \phi_2 : \mathcal{H}^2 \to \mathbb{R}^2$*, and* (ρ, η) *be half-space coordinates on* $\mathcal{H}^2$*, such that*

1. $\phi_1 \wedge \phi_2 > 0 \qquad\qquad \forall (\rho, \eta) \in \mathcal{H}^2$

2. $\frac{\partial \phi_1}{\partial \rho} + \frac{\partial \phi_2}{\partial \eta} = \frac{\phi_1}{\rho}$

3. $\frac{\partial \phi_1}{\partial \eta} - \frac{\partial \phi_2}{\partial \rho} = 0.$

Where here, and throughout, we have used the notation

$$\phi_1 \wedge \phi_2 = \det(\phi_1,\, \phi_2).$$

Then let $\psi_1,\, \psi_2 : \mathcal{H}^2 \to (\mathbb{R}^2)^*$ given by

$$\begin{aligned} \psi_1 &= -\frac{\det(\phi_2,\, .)}{\det(\phi_1,\, \phi_2)} \\ \psi_2 &= \frac{\det(\phi_1,\, .)}{\det(\phi_1,\, \phi_2)} \end{aligned}$$

so that

$$\phi_1 \otimes \psi_1 + \phi_2 \otimes \psi_2 = id \in \Gamma(\mathbb{R}^2 \otimes (\mathbb{R}^2)^*). \tag{4.2}$$

Identifying $\mathbb{R}^2$ with the tangent space to the torus,

$$g = \frac{d\rho^2 + d\eta^2}{\rho^2} + \psi_1^2 + \psi_2^2$$

is a self-dual metric on $U^2 \times T^2$.

Proof. Let $l_1,\, l_2$ be an orthonormal frame of L such that

$$\begin{aligned} l_1 \otimes l_1 - l_2 \otimes l_2 &\cong \rho\frac{\partial}{\partial \rho} \\ l_1 \otimes l_2 + l_2 \otimes l_1 &\cong \rho\frac{\partial}{\partial \eta}, \end{aligned}$$

with $\lambda_1,\, \lambda_2$ the basis dual to $l_1,\, l_2$ and

$$\omega_1 = \frac{1}{\rho} d\rho \qquad\qquad \omega_2 = \frac{1}{\rho} d\eta.$$

The identity section C is given by

$$C = \frac{1}{4}\left(\lambda_1 \otimes \rho\frac{\partial}{\partial \rho} \otimes \lambda_1 - \lambda_2 \otimes \rho\frac{\partial}{\partial \rho} \otimes \lambda_2 + \lambda_1 \otimes \rho\frac{\partial}{\partial \eta} \otimes \lambda_1 + \lambda_2 \otimes \rho\frac{\partial}{\partial \eta} \otimes \lambda_1\right).$$

Finally, let

$$\phi = \lambda_1 \otimes \phi_1 + \lambda_2 \otimes \phi_2.$$

Substituting this data in, the right-hand side of (4.1) is then

$$\frac{1}{2}\left(\lambda_1 \otimes \frac{1}{\rho} d\rho \otimes \phi_1 - \lambda_2 \otimes \frac{1}{\rho} d\rho \otimes \phi_2 + \right. \\ \left. + \lambda_1 \otimes \frac{1}{\rho} d\eta \otimes \phi_1 + \lambda_2 \otimes \frac{1}{\rho} d\eta \otimes \phi_2 \right). \tag{4.3}$$

The connection ∇^L is given by

$$\nabla^L l_1 = -\omega_1 \otimes \frac{l_2}{2} \qquad \nabla^L l_2 = \omega_2 \otimes \frac{l_1}{2}$$

so the dual connection has

$$\nabla^{L^*} \lambda_1 = -\omega_1 \otimes \frac{\lambda_2}{2} \qquad \nabla^{L^*} \lambda_2 = \omega_2 \otimes \frac{\lambda_1}{2},$$

so

$$\sum_{a,\beta} \nabla^L_a \phi_\beta = \left(\frac{\partial \phi_1}{\partial \rho} \otimes \lambda_1 \otimes d\rho + \frac{\partial \phi_2}{\partial \rho} \otimes \lambda_2 \otimes d\rho + \frac{\partial \phi_1}{\partial \eta} \otimes \lambda_1 \otimes d\eta + \right. \\ + \left. \frac{\partial \phi_2}{\partial \eta} \otimes \lambda_2 \otimes d\rho \right) + \left(\phi_1 \otimes \frac{\omega_1}{2} \otimes \lambda_2 + \phi_2 \otimes \frac{\omega_2}{2} \otimes \lambda_1 \right).$$

The complex structures are then given by

$$\begin{aligned} J(d\rho) &= d\eta \qquad J(d\eta) = -d\rho \\ \tilde{J}(\lambda_1) &= \lambda_2 \qquad \tilde{J}(\lambda_2) = -\lambda_1 \end{aligned}$$

so that the left-hand side

$$(d\rho \otimes \lambda_1 + d\eta \otimes \lambda_2)\left(\frac{\partial \phi_1}{\partial \rho} - \frac{\partial \phi_2}{\partial \eta}\right) + (d\rho \otimes \lambda_2 - d\eta \otimes \lambda_1)\left(\frac{\partial \phi_2}{\partial \rho} - \frac{\partial \phi_1}{\partial \eta}\right) + \\ - \left(\frac{\omega_2}{2} \otimes \lambda_2 - \frac{\omega_1}{2} \otimes \lambda_1\right)\phi_1 + \left(\frac{\omega_2}{2} \otimes \lambda_1 - \frac{\omega_1}{2} \otimes \lambda_2\right)\phi_2. \tag{4.4}$$

Equating (4.3) and (4.4) together, we are left with

$$(d\rho \otimes \lambda_1 + d\eta \otimes \lambda_2) \otimes \left(\frac{\partial \phi_1}{\partial \rho} + \frac{\partial \phi_2}{\partial \eta} - \rho \phi_1\right) + \\ + (d\rho \otimes \lambda_2 - d\eta \otimes \lambda_1)\left(\frac{\partial \phi_2}{\partial \rho} - \frac{\partial \phi_1}{\partial \eta}\right) = 0.$$

$\square$

Then in order to find a Joyce metric, it is sufficient to find a pair of $\mathbb{R}^2$ valued functions (ϕ_1, ϕ_2) satisfying these three conditions. We will refer to the latter two conditions as the *Joyce equations*, and to functions satisfying them as *Joyce solutions.*

Joyce then turns to finding solutions for these equations. Note that while ϕ_1, ϕ_2 are $\mathbb{R}^2$-valued, we can solve the equations for each component separately, so we will seek scalar solutions to the Joyce equations and then tensor these solutions with vectors in $\mathbb{R}^2$ to obtain vector solutions. Now, clearly $(\phi_1, \phi_2) = (0, 1)$ is a solution, so denote this solution

$$f^{(\infty)} = (0, 1).$$

We use this solution to generate a family of solutions — note that (4.1.1) is independent of our identification of $\mathcal{H}^2$ with the upper half plane and hence is invariant under isometries of $\mathcal{H}^2$. Then by acting on this solution with hyperbolic isometries we will be able to obtain new solutions.

If $\phi_1^{(i)}$, $\phi_2^{(i)}$, $i \leq n$ is a collection of scalar solutions of the Joyce equations and $v^{(i)} \in \mathbb{R}^2$, $i \leq n$, since the Joyce equations are linear equations,

$$(\phi_1, \phi_2) = \left(\sum_{i=1}^{n} \phi_1^{(i)} v^{(i)}, \sum_{i=1}^{n} \phi_2^{(i)} v^{(i)} \right)$$

gives a vector-valued solution, then we can check that condition (1) of (4.1.2) also holds for this sum, namely that

$$\phi_1 \wedge \phi_2 > 0,$$

and if this holds, we are able to apply (4.1.2) to obtain a self-dual metric.

In order to apply the hyperbolic isometries, we first require a more explicit model for $\partial\mathcal{H}^2$. This is considered in [23], and more explicitly in [5]. We can identify $\mathcal{H}^2$ with the group of positive definite symmetric matrices, up to a scale factor:

$$\eta + i\rho \mapsto \left[\begin{pmatrix} 1 & \eta \\ \eta & \rho^2 + \eta^2 \end{pmatrix} \right] \in \frac{\mathrm{Sym}(2)}{\mathbb{R}^+}.$$

This gives a natural model for the boundary as singular matrices,

$$y \mapsto \left[\begin{pmatrix} 1 & y \\ y & y^2 \end{pmatrix}\right] \in \frac{\mathrm{Sym}(2)}{\mathbb{R}^+}$$

with the other chart on $\partial\mathcal{H}^2 \cong \mathbb{R}P^1$ giving a similar expression around the point at infinity,

$$z \mapsto \left[\begin{pmatrix} z^2 & z \\ z & 1 \end{pmatrix}\right] \in \frac{\mathrm{Sym}(2)}{\mathbb{R}^+}.$$

Take a trivial $\mathbb{R}^2$-bundle, L. Then we can identify L^* with $\mathbb{R}^2$ by taking a basis

$$\lambda_1 = \frac{1}{\sqrt{\rho}}\begin{pmatrix} 0 \\ \rho \end{pmatrix} \qquad \lambda_2 = \frac{1}{\sqrt{\rho}}\begin{pmatrix} 1 \\ \eta \end{pmatrix}.$$

The dual basis to this then satisfies (4.2).

Using this model we can see that $\mathrm{SL}(2, \mathbb{R})$ acts on $\mathcal{H}^2$ by congruence,

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix} \cdot \left[\begin{pmatrix} 1 & \eta \\ \eta & \rho^2+\eta^2 \end{pmatrix}\right] = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \left[\begin{pmatrix} 1 & \eta \\ \eta & \rho^2+\eta^2 \end{pmatrix}\right] \begin{pmatrix} d & c \\ b & a \end{pmatrix}^T$$

which can be shown by direct calculation to correspond to a hyperbolic isometry

$$\tilde{\rho} + i\tilde{\eta} = \frac{a(\eta + i\rho) + b}{c(\eta + i\rho) + d}.$$

Then this action extends to the boundary,

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix} \cdot \begin{pmatrix} 1 & y \\ y & y^2 \end{pmatrix} = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} 1 & y \\ y & y^2 \end{pmatrix} \begin{pmatrix} d & c \\ b & a \end{pmatrix}^T$$

and induces an action on L^*,

$$\begin{aligned} \tilde{\lambda}_1 &= \tfrac{1}{\sqrt{\tilde{\rho}}}\begin{pmatrix} d & c \\ b & a \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \tilde{\rho} \end{pmatrix} \\ \tilde{\lambda}_2 &= \tfrac{1}{\sqrt{\tilde{\rho}}}\begin{pmatrix} d & c \\ b & a \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \tilde{\eta} \end{pmatrix}. \end{aligned} \tag{4.5}$$

Note that the initial action had centre $\pm I$, so this action descends to $\mathrm{PSL}(2, \mathbb{R})$ but the induced action on L^* does not. Then acting on our initial solution $f^{(\infty)}$ by hyperbolic isometries allows us to obtain a family of solutions, by applying these transformations with

$$\begin{pmatrix} d & c \\ b & a \end{pmatrix} = \begin{pmatrix} -y & 1 \\ -1 & 0 \end{pmatrix}$$

and equating

$$\phi = \tilde{\lambda}_1 \otimes \tilde{\phi}_1 + \tilde{\lambda}_2 \otimes \tilde{\phi}_2 = \lambda_1 \otimes \phi_1 + \lambda_2 \otimes \phi_2.$$

The solutions we obtain in this way (including $f^{(\infty)}$) we refer to as *basic solutions*,

$$\begin{array}{lll} f_1^{(\infty)} = 0 & f_1^{(y)} = \frac{\rho}{\sqrt{\rho^2+(\eta-y)^2}} & \\ & & y \in \mathbb{R}. \\ f_2^{(\infty)} = 1 & f_2^{(y)} = \frac{\eta-y}{\sqrt{\rho^2+(\eta-y)^2}} & \end{array}$$

Applying a dilation does not change the basic solution, so these are the only solutions which can be obtained in this way.

Example 4.1.3. *We can superpose these solutions, so for any $y_1, \ldots, y_n \in \partial\mathcal{H}^2$, $\underline{w}_1, \ldots, \underline{w}_n \in \mathbb{R}^2$,*

$$\phi_1 = \sum_{i=1}^{n} f_1^{(y_i)} \underline{w}_i \qquad\qquad \phi_2 \sum_{i=1}^{n} f_2^{(y_i)} \underline{w}_i$$

is a solution of the Joyce equations, and

$$g = \frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\det(\phi_1, d\underline{\theta})^2 + \det(\phi_2, d\underline{\theta})^2}{(\phi_1 \wedge \phi_2)^2}$$

is a self-dual metric on $\pi^{-1}(U)$, where

$$U = \{(\rho, \eta) \in \mathcal{H}^2 | \phi_1 \wedge \phi_2(\rho, \eta) > 0\}.$$

4.2 Extending to degenerate orbits

So far we have been able to construct self-dual metrics on $\mathcal{H}^2 \times T^2$. Following [23], we now take toric manifolds with orbit space given by diagrams as in (3.3.1) — identifying the interior of the orbit space with $\mathcal{H}^2$ we obtain a self-dual metric on a dense open set, and examine the question of when this metric can be extended to the degenerate orbits. We use the convolution notation of [5] here, as this can be more easily generalised to later results.

We begin by looking at the asymptotic behaviour of the solutions so far found, and then proceed to show that given this behaviour close to $\partial\mathcal{H}^2$, for appropriate choices of conformal factor, as we approach the degenerate orbits the metric we obtain behaves either like a neighbourhood of an axis, or a neighbourhood of the origin, in $\mathbb{R}^4$ in the standard metric. Using this comparison we conclude that the metrics extend to a C^2 metric over these orbits.

Consider a linear combination of basic solutions,

$$\phi = \sum_{j=0}^{n} f^{(y_k)} \otimes \underline{u}_k + f^{(\infty)} \otimes \underline{u}_\infty \qquad \underline{u}_k \in \mathbb{R}^2.$$

We can express ϕ as a convolution of $f(y) = f^{(y)}$ with a compactly supported distribution,

$$\begin{aligned} \phi_1 &= \int \tfrac{\rho}{\sqrt{\rho^2+(\eta-y)^2}} \underline{u}(y)\,dy \\ \phi_2 &= \int \tfrac{\eta-y}{\sqrt{\rho^2+(\eta-y)^2}} \underline{u}(y)\,dy + \underline{u}_\infty \end{aligned} \tag{4.6}$$

where

$$\underline{u}(y) = \sum_{k=0}^{n} \delta_{y_k} \otimes \underline{u}_k,$$

and δ_p is the Dirac delta function at p. Now let $\underline{w} : \mathbb{R} \to \mathbb{R}^2$ be a step function such that

$$\begin{aligned} \frac{d\underline{w}}{dx} &= \underline{u} \\ \lim_{x\to\infty} \underline{w}(x) &= \sum_{k=0}^{n} \underline{u}_k + \underline{u}_\infty. \end{aligned}$$

We call this $\underline{w}$ the *boundary data function.* Then integrating by parts gives us

$$\begin{aligned} \phi_1 &= \int \frac{\rho(y-\eta)}{(\rho^2+(\eta-y)^2)^{\frac{3}{2}}} \underline{w}(y)\,dy \\ \phi_2 &= \int \frac{\rho^2}{(\rho^2+(\eta-y)^2)^{\frac{3}{2}}} \underline{w}(y)\,dy. \end{aligned}$$

Finally, we must deal with the point at infinity. In order to do this, we use the argument of [5] regarding isometries of the underlying hyperbolic space to provide a change of coordinates. This will transform this point to the origin and allow us to apply the same results here.

Proposition 4.2.1. *[5] Take a sum of basic solutions ϕ with boundary data function $\underline{w}$. Applying the change of coordinates*

$$(\tilde{\rho}, \tilde{\eta}) = \left(\frac{\rho}{\rho^2+\eta^2}, -\frac{\eta}{\rho^2+\eta^2}\right)$$

to $\mathcal{H}^2$ transforms this to another sum of basic solutions with boundary data

$$\underline{\tilde{w}}(y) = \operatorname{sign}(y)\underline{w}\left(-\frac{1}{y}\right).$$

Proof. ϕ is invariant under this change of coordinates, but we must change basis in L^* so equating the two expressions gives us

$$\phi = \lambda \otimes \phi_1 + \lambda_2 \otimes \phi_2 = \tilde{\lambda}_1 \otimes \tilde{\phi}_1 + \tilde{\lambda}_2 \otimes \tilde{\phi}_2.$$

Substituting

$$\lambda_1 = \tfrac{1}{\sqrt{\rho}}\begin{pmatrix} 0 \\ \rho \end{pmatrix} \qquad \lambda_2 = \tfrac{1}{\sqrt{\rho}}\begin{pmatrix} 1 \\ \eta \end{pmatrix}$$

$$\tilde{\lambda}_1 = \tfrac{\sqrt{\rho^2+\eta^2}}{\sqrt{\rho}}\begin{pmatrix} -\frac{\rho}{\rho^2+\eta^2} \\ 0 \end{pmatrix} \qquad \tilde{\lambda}_2 = \tfrac{\sqrt{\rho^2+\eta^2}}{\sqrt{\rho}}\begin{pmatrix} \frac{\eta}{\rho^2+\eta^2} \\ 1 \end{pmatrix}$$

as in (4.5) gives us

$$\begin{pmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \end{pmatrix} = \frac{1}{\sqrt{\rho^2+(\eta-y)^2}}\begin{pmatrix} \eta-y & -\rho \\ \rho & \eta-y \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

Using the change of variables $\tilde{y} = -\frac{1}{y}$ and the formula

$$\sqrt{\tilde{\rho}^2+(\tilde{\eta}-\tilde{y})^2} = \frac{\sqrt{\rho^2+(\eta-y)^2}}{|y|\sqrt{\rho^2+\eta^2}}$$

we can show directly that this is equal to the proposed solution,

$$\begin{aligned} \tilde{\phi}_1 &= \int \frac{\tilde{\rho}(\tilde{y}-\tilde{\eta})}{(\tilde{\rho}^2+(\tilde{\eta}-\tilde{y})^2)^{\frac{3}{2}}}\underline{\tilde{w}}(\tilde{y})\, d\tilde{y} \\ \tilde{\phi}_2 &= \int \frac{\tilde{\rho}^2}{(\tilde{\rho}^2+(\tilde{\eta}-\tilde{y})^2)^{\frac{3}{2}}}\underline{\tilde{w}}(\tilde{y})\, d\tilde{y} \end{aligned}$$

□

This result tells us that if we identify $\partial\mathcal{H}^2$ with $\mathbb{R}P^1$ by

$$(0,\, y) \in \partial\mathcal{H}^2 \mapsto [(1,\, y)] \in \mathbb{R}P^1, \tag{4.7}$$

then $\underline{w}$ can be viewed invariantly as an $\mathbb{R}^2$-valued function on the tautological line bundle with the property that

$$\underline{w}\left([1,\, y],\, (\lambda,\, \lambda y)\right) = \mathrm{sign}(\lambda)\underline{w}\left([1,\, y],\, (1,\, y)\right). \tag{4.8}$$

This allows us to extend what follows to the point at infinity by applying the same results to the transformed solution.

Following [5], we show that the boundary data function $\underline{w}$ gives the asymptotic behaviour of $(\phi_1,\, \phi_2)$. We shall see later (4.2.5) that these asymptotic values correspond to the weighting of the arcs in the Orlik-Raymond diagram (3.8) of the manifold over which the metric is defined. Now, suppose that this satisfies

$$\det(\underline{w}(y),\, \underline{w}(z)) \leq 0 \qquad \forall y \leq z, \tag{4.9}$$

and that near each discontinuity y_k it has the form

$$\underline{w}(y) = \begin{cases} (m,\, n) & y_k - \delta < y < y_k \\ (m',\, n') & y_k < y < y_k + \delta \end{cases} \tag{4.10}$$

with $\det \begin{pmatrix} m & m' \\ n & n' \end{pmatrix} = -1$ and that it has at least one discontinuity y_k. Then

$$\begin{aligned} \phi_1 &= \int \frac{\rho(y-\eta)}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}}\, \underline{w}(y)dy = \sum \frac{\rho}{\sqrt{\rho^2 + (\eta - y)^2}}\, \underline{u}_k \\ \phi_2 &= \int \frac{\rho^2}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}}\, \underline{w}(y)dy = \sum \frac{\eta - y}{\sqrt{\rho^2 + (\eta - y)^2}}\, \underline{u}_k + \underline{u}_\infty \end{aligned}$$

with

$$\underline{u}_k = \frac{(m - m', n - n')}{2}.$$

The first condition here ensures that the given boundary data is that of a polytope, with a consistent choice of inward-pointing normal, while the second ensures that each of the vertices (except possibly the point at infinity, which will need to be dealt with separately) satisfy the Delzant condition.

We refer to a boundary point at which the boundary data function is discontinuous as a *vertex*, and any other boundary point as an *edge point*. We say that a boundary data function satisfying (4.9) is *convex*. If the boundary data function satisfies (4.10) at a vertex, we say the vertex is *non-singular*, and if this is true of all the vertices we say the boundary data is *non-singular*. We show that close to the boundary points such a metric approaches that of a point on the axis, or the origin, in $\mathbb{R}^4$, in the standard metric, considered in polar coordinates. We then use this approximation to extend the metric to the degenerate orbits.

Lemma 4.2.2. *[23] For a finite sum of basic solutions satisfying (4.9) and (4.10),*

- *near any boundary point $z \neq y_0, \ldots, y_n$ there is a neighbourhood on which*

$$\begin{aligned} \phi_1(\rho, \eta) &= O(\rho) \\ \phi_2(\rho, \eta) &= \underline{w}(z) + O(\rho^2). \end{aligned}$$

- *and near any vertex $y = y_i$ if $r = \sqrt{\rho^2 + (\eta - y_i)^2}$,*

$$\begin{aligned} \phi_1(\rho, \eta) &= \frac{\rho}{2r}((m, n) - (m', n')) + O(r) \\ \phi_2(\rho, \eta) &= \frac{1}{2}((m, n) + (m', n')) + \frac{\eta - y}{r}\frac{1}{2}((m, n) - (m', n')) + O(r^2) \end{aligned}$$

for some (m, n), (m', n') with $\det \begin{pmatrix} m & m' \\ n & n' \end{pmatrix} = -1.$

Proof. • Since $|y - y_k|$ is bounded away from zero, for sufficiently small ρ we have $|\frac{\rho}{\eta - y_k}| < 1$. Then we can take a binomial expansion of the denominators,

$$\frac{1}{r} = \frac{1}{|\eta - y_k|} + O(\rho^2).$$

Substituting these expressions into

$$\begin{aligned} \phi_1 &= \int \frac{\rho}{\sqrt{\rho^2 + (\eta - y)^2}} \underline{u}(y)\, dy \\ \phi_2 &= \int \frac{\eta - y}{\sqrt{\rho^2 + (\eta - y)^2}} \underline{u}(y)\, dy, \end{aligned}$$

and noting that $\underline{u}$ is zero for y close to z gives the result.

- Applying the first part of this lemma to $\phi - f^{(y_k)} \otimes \underline{u}_k$ we see ϕ must have the form

$$\begin{aligned}\phi_1 &= \frac{\rho}{r}\underline{v}_1 + O(\rho)\\ \phi_2 &= \frac{\eta - y}{r}\underline{v}_1 + \underline{v}_2 + O(\rho^2).\end{aligned}$$

Comparing this with the boundary data function, for some $\epsilon > 0$

$$\lim_{\rho\to 0}\phi_2\big|_{(y-\epsilon, y+\epsilon)} = \begin{cases} \underline{v}_2 - \underline{v}_1 & \eta < y \\ \underline{v}_2 + \underline{v}_1 & \eta > y \end{cases} = \begin{cases} (m,\, n) & \eta < y \\ (m',\, n') & \eta > y \end{cases}$$

and solving this gives

$$\begin{aligned}\underline{v}_1 &= \frac{(m',\, n') - (m,\, n)}{2}\\ \underline{v}_2 &= \frac{(m',\, n') + (m,\, n)}{2}.\end{aligned}$$

Then note that $\rho < r$, so that we can express the error terms in r and this gives the result.

□

We show that for such a solution $\phi_1 \wedge \phi_2$ is positive — in fact, we can do slightly better, using the asymptotic behaviour to extend this to the boundary as follows: We have seen that $\phi_1 = O(\rho)$ on boundary points other than the vertices, so we can define $\phi_1' = \frac{1}{\rho}\phi_1$ and this section extends to $\bar{\mathcal{H}}^2 \setminus \{y_0, \ldots, y_n\}$.

Proposition 4.2.3. *[23] For a finite sum of basic solutions satisfying (4.9) and (4.10),*

$$\phi_1' \wedge \phi_2 > 0 \qquad \forall(\rho,\, \eta) \in \bar{\mathcal{H}}^2 \setminus \{y_0, \ldots, y_n\}.$$

Proof. For any interior point $(\rho,\, \eta) \in \mathcal{H}^2$,

$$\begin{aligned}\phi_1' \wedge \phi_2 &= \int\!\!\int \frac{\rho^2(y-\eta)}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}(\rho^2 + (\eta - z)^2)^{\frac{3}{2}}} \det(\underline{w}(y),\, \underline{w}(z))\, dydz\\ &= \int\!\!\int_{y\le z} \frac{\rho^2(y-\eta) - \rho^2(z-\eta)}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}(\rho^2 + (\eta - z)^2)^{\frac{3}{2}}} \det(\underline{w}(y),\, \underline{w}(z))\, dydz\\ &= \int\!\!\int_{y\le z} \frac{\rho^2(y-z)}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}(\rho^2 + (\eta - z)^2)^{\frac{3}{2}}} \det(\underline{w}(y),\, \underline{w}(z))\, dydz.\end{aligned}$$

Since the integrand is everywhere non-negative, and since we have at least one vertex, it is strictly positive on some rectangle. Therefore $\phi_1' \wedge \phi_2 > 0$.

Now consider a boundary point $(0,\, \eta_0)$ with $\eta_0 \neq y_0, \ldots,\, y_n$. Then for some $\delta > 0$, $\underline{w}$ is constant on $(\eta_0 - \delta,\, \eta_0 + \delta)$ and suppose there is some $\eta_0 < a < b$ such that

$$\det(\underline{w}(\eta_0),\, \underline{w}(z)) \leq -1 \qquad\qquad \text{for almost all } a < z < b.$$

Since $\underline{w}$ is convex,

$$\begin{aligned}
\phi_1' \wedge \phi_2(\rho,\, \eta_0) \quad &\geq \quad \left(\int_a^b \int_{\eta_0-\delta}^{\eta_0+\delta} - \int_{\eta_0-\delta}^{\eta_0+\delta} \int_a^b \right) \\
&\qquad \frac{\rho^2 (z-\eta_0)}{(\rho^2 + (\eta_0 - y)^2)^{\frac{3}{2}} (\rho^2 + (\eta_0 - z)^2)^{\frac{3}{2}}} \det(\underline{w}(z),\, \underline{w}(\eta_0))\, dydz \\
&\geq \quad \left(\int_{\eta_0-\delta}^{\eta_0+\delta} \frac{\rho^2}{(\rho^2 + (\eta_0 - y)^2)^{\frac{3}{2}}}\, dy \right) \left(\int_a^b \frac{z-\eta_0}{(\rho^2 + (\eta_0 - z)^2)^{\frac{3}{2}}}\, dz \right) + \\
&\qquad - \left(\int_{\eta_0-\delta}^{\eta_0+\delta} \frac{\rho^2}{(\rho^2 + (\eta_0 - z)^2)^{\frac{3}{2}}}\, dz \right) \left(\int_a^b \frac{y-\eta_0}{(\rho^2 + (\eta_0 - y)^2)^{\frac{3}{2}}}\, dy \right).
\end{aligned}$$

Each of these integrals is bounded, and the third is zero by symmetry, so we are left with just the first term:

$$\begin{aligned}
\phi_1' \wedge \phi_2(\rho,\, \eta_0) \quad &\geq \quad 2 \frac{\delta}{\sqrt{\rho^2 + (\eta_0 - \delta)^2}} \int_a^b \frac{z - \eta_0}{(\rho^2 + (\eta_0 - z)^2)^{\frac{3}{2}}}\, dz \\
&\xrightarrow{\rho \to 0} \quad 2 \int_a^b \frac{\eta - z}{|\eta - z|^3}\, dz.
\end{aligned}$$

In particular this is positive, so $\phi_1' \wedge \phi_2(0,\, \eta_0)$ is bounded away from 0.

If not, since $\underline{w}$ has at least one vertex there are $a < b < \eta_0$ with

$$\det(\underline{w}(z),\, \underline{w}(\eta_0)) \leq -1 \qquad\qquad \text{for almost all } a < z < b,$$

and the same proof holds with the determinant and the $(z - \eta_0)$ in the numerator both negative. $\square$

We shall take this approach further in section (7.1), where we will calculate this asymptotic value exactly and remove the assumption of convexity.

This estimate allows us to compare the Joyce metrics near edge points with the standard metric on $\mathbb{R}^4$ near an axis. We will also wish to compare fixed points with the origin of $\mathbb{R}^4$, for which we will need to express this metric in polar form.

Lemma 4.2.4. *[23] Let $(r_1,\, \theta_1,\, r_2,\, \theta_2)$ be radial coordinates on $\mathbb{R}^2 \times \mathbb{R}^2$, and identify the quadrant $r_1,\, r_2 > 0$ with the upper half plane $\mathcal{H}^2$ by*

$$\eta + i\rho = (r_1 + ir_2)^2.$$

In these coordinates the flat metric is

$$g_{flat} = \frac{d\rho^2 + d\eta^2}{4\sqrt{\rho^2+\eta^2}} + \frac{1}{2}(\sqrt{\rho^2+\eta^2} - \eta)d\theta_1^2 + \frac{1}{2}(\sqrt{\rho^2+\eta^2} + \eta)d\theta_2^2$$

Then suppose we have a solution ϕ of the Joyce equations (4.1.2) with $\phi_1 \wedge \phi_2 > 0$, and that M is a toric 4-manifold with boundary data function $\underline{w}$ with vertices $y_0, \ldots,\, y_n$, that near any point $(0,\, y) \neq (0,\, y_1), \ldots,\, (0, y_n)$,

$$\left.\begin{array}{l} \phi_1(\rho,\, \eta) = O(\rho) \\ \phi_2(\rho,\, \eta) = \underline{w}(\eta) + O(\rho^2) \\ \phi_1' = \frac{\phi_1}{\rho} \text{ can be extended to the boundary, and then } \phi_1' \wedge \phi_2 > 0. \end{array}\right\} \tag{4.11}$$

and that near any point $(0,\, y_i)$, if $r = \sqrt{\rho^2 + (\eta - y_i)^2}$,

$$\left.\begin{array}{l} \phi_1(\rho,\, \eta) = \frac{\rho}{2r}((m,\, n) - (m',\, n')) + O(r) \\ \phi_2(\rho,\, \eta) = \frac{1}{2}((m,\, n) + (m',\, n')) + \frac{\eta - y}{r}\frac{1}{2}((m,\, n) - (m',\, n')) + O(r^2). \end{array}\right\} \tag{4.12}$$

Theorem 4.2.5. *[23] If the conformal factor Ω^2 is chosen such that*

$$\left.\begin{array}{ll} \textit{if } y \neq y_0, \ldots,\, y_n, & \frac{\Omega^2}{\rho^2} \textit{ is } C^2 \textit{ and positive near } (0,\, y) \textit{ in } \bar{\mathcal{H}}^2 \\ \textit{if } y = y_j, & \frac{\sqrt{\rho^2 + (\eta - y_j)^2}\Omega^2}{\rho^2} \textit{ is } C^2 \textit{ and positive near } (0,\, y_j) \textit{ in } \bar{\mathcal{H}}^2, \end{array}\right\} \tag{4.13}$$

and $\phi_1,\, \phi_2$ satisfy (4.11) and (4.12) then the Joyce metric (4.1.2)

$$g = \Omega^2\left(\frac{d\rho^2 + d\eta^2}{\rho^2} + \psi_1^2 + \psi_2^2\right)$$

extends to a C^2 self-dual metric on $M \setminus \pi^{-1}((0,\, \infty))$.

Proof. Consider first a point of the first kind, with stabiliser $G(m,\, n)$, and a neighbourhood U of $(0,\, y)$ in $\bar{\mathcal{H}}^2$ containing no fixed points.

Let $\det \begin{pmatrix} m & m' \\ n & n' \end{pmatrix} = -1$ and $z_1 \equiv n\theta_1 - m\theta_2$, $z_2 \equiv n'\theta_1 - m'\theta_2$, where θ_1, θ_2 are coordinates on T^2, and both equations are up to multiples of 2π. Now, let $\psi_1' = \rho\psi_1$, so that

$$\phi_1' \otimes \psi_1' + \phi_2 \otimes \psi_2 = \mathrm{id}$$

so using the asymptotics above,

$$\begin{aligned} \psi_1'((m,\, n)) &= \rho\psi_1(\phi_2 + O(\rho^2)) = O(\rho^2) \\ \psi_2((m,\, n)) &= 1 + O(\rho^2) \end{aligned}$$

and hence

$$\begin{aligned} \psi_1' &= \rho s_1 dz_1 + O(\rho^2) dz_2 \\ \psi_2 &= s_2 dz_1 + (1 + O(\rho^2)) dz_2 \end{aligned}$$

for some functions s_1, s_2, and $s_1(0,\, \eta) \neq 0$. Then the metric becomes

$$\begin{aligned} g &= \frac{\Omega^2}{\rho^2}(d\rho^2 + d\eta^2 + (\psi_1')^2 + \rho^2\psi_2^2) \\ &= \frac{\Omega^2}{\rho^2}(d\rho^2 + (1 + O(\rho^2))dz_2^2 + d\eta^2 + \\ &\qquad +(\rho^2 s_1^2 + O(\rho^2))dz_1^2 + O(\rho^2)dz_1 dz_2). \end{aligned}$$

Since U contains no fixed points, $\pi^{-1}(U)$ is diffeomorphic to a neighbourhood of the axis $r_1 = 0$ in $\mathbb{R}^4$. Here the flat metric is

$$g_{\text{flat}} = dr_1^2 + r_1^2 d\theta_1^2 + dr_2^2 + r_2^2 d\theta_2^2.$$

Comparing the two metrics we can see that provided $\frac{\Omega^2}{\rho^2}$ is smooth and positive, g extends to a C^2 metric on $\pi^{-1}(U)$.

Similarly, if $(0,\, y)$ is a vertex, let U be a neighbourhood in $\bar{\mathcal{H}}^2$ containing only this vertex. As before, using the asymptotic behaviour of $(\phi_1,\, \phi_2)$ from (4.12) we can deduce that

$$\begin{aligned}
\psi_1\left(\frac{\rho}{2r}(m-m',\, n-n')\right) &= 1+O(r)\\
\psi_2\left(\frac{\rho}{2r}(m-m',\, n-n')\right) &= O(r)\\
\psi_1\left(\frac{1}{2}(m+m',\, n+n')+\frac{\eta-y}{2r}(m-m',\, n-n')\right) &= O(r^2)\\
\psi_2\left(\frac{1}{2}(m+m',\, n+n')+\frac{\eta-y}{2r}(m-m',\, n-n')\right) &= 1+O(r^2).
\end{aligned}$$

Then if we set

$$\begin{aligned}
z_1 &= m\theta_2 - n\theta_1\\
z_2 &= m'\theta_2 - n'\theta_1
\end{aligned}$$

by evaluating ψ_1, ψ_2 on $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$, we find that

$$\begin{aligned}
\psi_1 &= \frac{(\eta-y)+r}{\rho}dz_1 + \frac{r-(\eta-y)}{\rho}dz_2 + O(r)\\
\psi_2 &= dz_1 + dz_2 + O(r).
\end{aligned}$$

Substituting these expressions into g and comparing with (4.2.4) we find that

$$\begin{aligned}
g &= \frac{r\Omega^2}{\rho^2}\left(\frac{d\rho^2+d\eta^2}{\sqrt{\rho^2+(\eta-y)^2}} + 2(r+(\eta-y))dz_1^2 + \right.\\
&\qquad \left. +2(r-(\eta-y))dz_2^2 + \frac{\rho^2}{r}O(r)\right)\\
&= 4\frac{r\Omega^2}{\rho^2}\left(g_{\text{flat}}+O(r^2)\right),
\end{aligned}$$

so that g extends to a C^2 metric over the vertex. $\square$

Then to build families of self-dual metrics on a symplectic toric manifold we need only find a Joyce solution with the appropriate boundary data. Joyce [23] does this as follows:

Theorem 4.2.6. *([23], [5]) Let M be a compact symplectic toric manifold. The orbit space, being a compact polytope, is simply connected, and we identify it with $\bar{\mathcal{H}}^2$. If*

$$y_n < \ldots < y_0 \in \partial\mathcal{H}^2$$

and possibly ∞ *are the fixed points of the action and* $G(m_i, n_i)$ *the stabilisers of the arc* (y_i, y_{i-1}), $0 \le i \le k+1$, *where for simplicity we write* $y_{-1} = \infty$, $y_{k+1} = -\infty$, *and if*

- $\det \begin{pmatrix} m_{j+1} & m_j \\ n_{j+1} & n_j \end{pmatrix} = -1$

- $\det \begin{pmatrix} m_k & m_j \\ n_k & n_j \end{pmatrix} \le 0 \quad \forall j \le k,$

that is, the boundary data is non-singular and convex, then for appropriate choices of conformal factor, the Joyce metric given by the Joyce solution

$$\phi = \frac{1}{2}\sum_{i=1}^{k} f^{(y_i)} \otimes (m_{i-1} - m_i,\, n_{i-1} - n_i) + \frac{1}{2} f^{(\infty)} \otimes (m_0 + m_{k+1},\, n_0 + n_{k+1})$$

extends to $M \setminus \pi^{-1}((0, \infty))$.

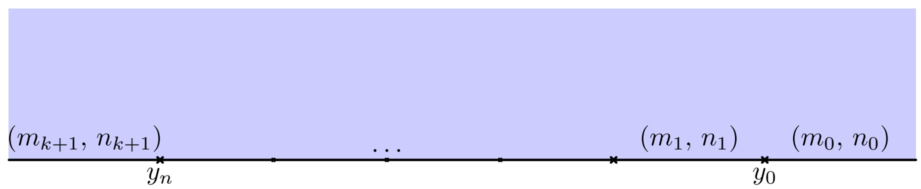


Figure 4.1: Boundary data for M.

Proof. From (4.2.2) we know that near a boundary point $y_k < y < y_{k+1}$,

$$\begin{aligned} \phi_1 &= O(\rho) \\ \phi_2 &= (m_k, n_k) + O(\rho^2), \end{aligned}$$

and we have seen (4.2.3) that for such a sum of basic solutions,

$$\phi_1' \wedge \phi_2 > 0 \quad \forall (\rho, \eta) \in \bar{\mathcal{H}}^2 \setminus \{y_0, \dots, y_n\}.$$

Then by (4.2.5) the Joyce metric extends as required. ☐

Corollary 4.2.7. *[5] If, furthermore, with the notation of the previous theorem*

$$\det \begin{pmatrix} m_0 & -m_{k+1} \\ n_0 & -n_{k+1} \end{pmatrix} = -1$$

or

$$(m_0,\, n_0) = -(m_{k+1},\, n_{k+1})$$

then the conformal metric extends to all of M. We will say such boundary data is non-singular and odd at infinity.

Proof. By (4.2.1) these conditions are precisely what is needed to ensure that when we apply a hyperbolic inversion to the boundary data, the point at infinity be either a non-singular fixed point or an edge point with stabiliser $G(m_0,\, n_0)$ respectively. Then using this chart we can see that the conformal metric extends to this orbit too. □

Example 4.2.8.

- $\mathbb{C}P^2$. *Using the Orlik-Raymond diagram (3.3.2), we need only choose positions for the vertices and signs for the stabilisers satisfying (4.2.6) to obtain a boundary data function*

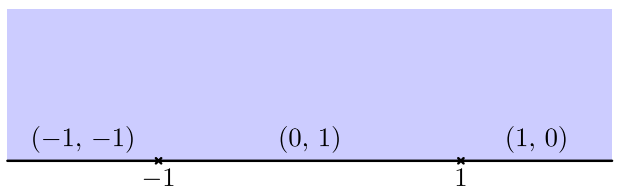


Figure 4.2: Boundary data for $\mathbb{C}P^2$.

The corresponding Joyce solution is

$$\begin{aligned}
\phi_1 &= \frac{\rho}{2\sqrt{\rho^2 + (\eta - 1)^2}}(1,\, -1) + \frac{\rho}{2\sqrt{\rho^2 + (\eta + 1)^2}}(1,\, 2) \\
\phi_2 &= \frac{\eta - 1}{2\sqrt{\rho^2 + (\eta - 1)^2}}(1,\, -1) + \frac{\eta + 1}{2\sqrt{\rho^2 + (\eta + 1)^2}}(1,\, 2) + \frac{1}{2}(0,\, -1).
\end{aligned}$$

Note that $\det((-1,\, -1),\, (1,\, 0)) = -1$ *so this metric also extends to the point at infinity. Then the Joyce metric for this solution gives a self-dual conformal metric on* $\mathbb{C}P^2$.

- $S^2 \times \mathcal{H}^2$. *Again we can choose boundary data satisfying (4.2.6),*

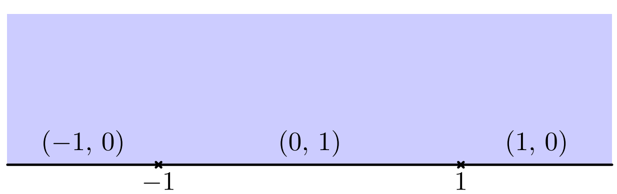


Figure 4.3: Boundary data for $S^2 \times \mathcal{H}^2$.

giving a Joyce solution

$$\begin{aligned}\phi_1 &= \frac{\rho}{2\sqrt{\rho^2+(\eta-1)^2}}(1,\,-1)+\frac{\rho}{2\sqrt{\rho^2+(\eta+1)^2}}(1,\,1)\\ \phi_2 &= \frac{\eta-1}{2\sqrt{\rho^2+(\eta-1)^2}}(1,\,-1)+\frac{\eta+1}{2\sqrt{\rho^2+(\eta+1)^2}}(1,\,1),\end{aligned}$$

so this gives a Joyce metric on $S^2 \times \mathcal{H}^2$*.*

- $\mathbb{C}^2$ *and* S^4*. From the Orlik-Raymond diagram for* $\mathbb{C}^2$ *we can pick boundary data*

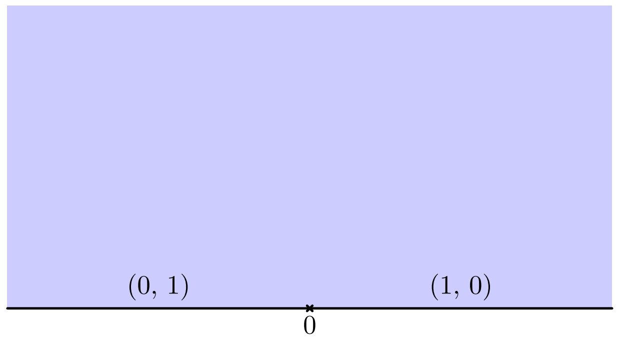


Figure 4.4: Boundary data for $\mathbb{C}$ and S^4.

with Joyce solution

$$\begin{aligned}\phi_1 &= \frac{\rho}{2\sqrt{\rho^2+\eta^2}}(1,\,-1)\\ \phi_2 &= \frac{\eta}{2\sqrt{\rho^2+\eta^2}}(1,\,-1)+\frac{1}{2}(1,\,1).\end{aligned}$$

This then gives us a self-dual conformal metric on $\mathbb{C}^2$*. Since*

$$\det((0,\,1),\,(1,\,0)) = -1$$

this conformal metric extends to the compactification S^4*.*

Chapter 5

Families of Joyce metrics

In this chapter we apply the Joyce construction to find specific families of metrics. We first see that for a family of conformal factors, Ω^2, the Joyce metrics are Kähler metrics of zero scalar curvature, as shown by Joyce [23]. We then discuss a result of Calderbank-Pedersen [4], which shows that subject to a linear constraint on the solution there is also a choice of conformal factor for which the Joyce metric is Einstein. We describe a construction of Calderbank-Singer [5] which encodes this data into a continued fraction expansion of a rational $q \in (0, 1)$.

We demonstrate results from [5] which show that for Joyce solutions given as sums of basic solutions with convex, non-singular boundary data these conformal factors satisfy the asymptotic conditions (4.13) on $\partial\mathcal{H}^2 \setminus \{(0, \infty)\}$, so that the metrics extend to the degenerate orbits.

We examine further results in [5] which extend these constructions to show that taking integrals rather than sums of basic solutions to give a larger family of solutions on smaller, non-compact regions. We look to some further extensions to these constructions — in [23], Joyce constructs self-dual metrics with fundamental group $\mathbb{Z}$ as quotients of the original Joyce metrics, and in [6] Calderbank-Singer extend the continued fraction construction to include irrationals, thereby obtaining Einstein metrics on manifolds of infinite topological type.

5.1 Scalar flat Kähler metrics

We quote without proof a result which tells us that for a family of conformal factors the representative of the Joyce class is a Kähler metric of zero scalar curvature on $\mathcal{H}^2 \times T^2$.

This result is proven by Joyce in [23], but here we refer to the more explicit statement in [5]. We show that for a sum of basic solutions this conformal factor satisfies the asymptotic conditions (4.13) and hence the metric extends to $\pi^{-1}(\bar{\mathcal{H}}^2 \setminus \{(0, \infty)\})$.

We quote the following result from [5]:

Theorem 5.1.1. *[23] Let (ϕ_1, ϕ_2) be a solution of the Joyce equations. The representative of the Joyce class*

$$g = \rho\phi_1 \wedge \phi_2 \left(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\det(\phi_1, \cdot)^2 + \det(\phi_2, \cdot)^2}{(\phi_1 \wedge \phi_2)^2} \right) \tag{5.1}$$

is a scalar flat Kähler metric on $\mathcal{H}^2 \times T^2$, as is

$$g_y = \frac{\rho\phi_1 \wedge \phi_2}{\sqrt{\rho^2 + (\eta - y)^2}} \left(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\det(\phi_1, \cdot)^2 + \det(\phi_2, \cdot)^2}{(\phi_1 \wedge \phi_2)^2} \right) \tag{5.2}$$

for any $y \in \mathbb{R} \subseteq \partial\mathcal{H}^2$.

Joyce demonstrates [23] that the condition for the first conformal factor to give a scalar flat Kähler metric is equivalent to ϕ satisfying the Joyce equations. The other metrics are then obtained by applying a hyperbolic isometry to the underlying space, giving us for each Joyce solution ϕ a family of scalar flat Kähler metrics parametrised by $\partial\mathcal{H}^2$.

Now suppose that ϕ is a sum of basic solutions and satisfies (4.9) and (4.10). Having studied the asymptotic behaviour of such solutions we are able to show this choice of conformal factor satisfies the conditions (4.13),

Proposition 5.1.2. *[23] If ϕ is a sum of basic solutions satisfying (4.9) and (4.10), the conformal factor*

$$\Omega^2 = \rho\phi_1 \wedge \phi_2$$

satisfies the asymptotic conditions (4.13) at all points in $\partial\mathcal{H}^2 \setminus \{(0, \infty)\}$.

Proof. If $\phi = \sum_{i=0}^{n} f^{(y_i)} \otimes u_i$, for any point $y \neq y_0, \ldots, y_n$ we know from (4.2.5) that there is a neighbourhood $U \subseteq \bar{\mathcal{H}}^2$ on which $\phi_1' \wedge \phi_2$ is smooth and positive. But

$$\rho^{-2}\Omega^2 = \frac{1}{\rho}\phi_1 \wedge \phi_2 = \phi_1' \wedge \phi_2$$

so this extends smoothly and positively.

Likewise, near any vertex y_i there is a neighbourhood on which

$$\begin{aligned}
\phi_1 &= \frac{\rho}{\sqrt{\rho^2+(\eta-y_i)^2}} \otimes \frac{1}{2}((m,\,n)-(m',\,n')) + O(\rho) \\
\phi_2 &= \frac{1}{2}((m,\,n)+(m',\,n')) + \frac{\eta-y_i}{\sqrt{\rho^2+(\eta-y_i)^2}} \otimes \\
&\qquad \otimes \frac{1}{2}((m,\,n)-(m',\,n')) + O(\rho^2)
\end{aligned}$$

and it follows that

$$\begin{aligned}
\sqrt{\rho^2+(\eta-y_i)^2}\rho^{-2}\Omega^2 &= \frac{\sqrt{\rho^2+(\eta-y_i)^2}}{\rho}\phi_1\wedge\phi_2 \\
&= \frac{\sqrt{\rho^2+(\eta-y_i)^2}}{\rho}\left(\frac{\rho}{2\sqrt{\rho^2+(\eta-y_i)^2}} + O(\rho^2)\right)
\end{aligned}$$

is C^2 and positive. □

Example 5.1.3. • $\mathbb{C}^2$. *If we apply this choice of conformal factor to the Joyce metric found for $\mathbb{C}^2$ in (4.2.8),*

$$\Omega^2 = \rho\phi_1\wedge\phi_2 = \frac{\rho^2}{\sqrt{\rho^2+\eta^2}}.$$

Setting $r=\sqrt{\rho^2+\eta^2}$ then gives

$$\begin{aligned}
g_J &= \frac{d\rho^2+d\eta^2}{2r} + (r-\eta)d\theta_1^2 + (r+\eta)d\theta_2^2 \\
&= 2g_{flat}
\end{aligned}$$

using the form of the flat metric given in (4.2.4).

- $S^2\times\mathcal{H}^2$. *This time we have*

$$\Omega^2 = \rho\phi_1\wedge\phi_2 = \frac{\rho^2}{\sqrt{\rho^2+(\eta-1)^2}\sqrt{\rho^2+(\eta+1)^2}}$$

and a Joyce metric

$$\begin{aligned}
g_J &= \frac{d\rho^2+d\eta^2}{\sqrt{\rho^2+(\eta-1)^2}\sqrt{\rho^2+(\eta+1)^2}} + \\
&\qquad + \sqrt{\rho^2+(\eta-1)^2}\sqrt{\rho^2+(\eta+1)^2}\left(\det(\phi_1,\,\cdot)^2+\det(\phi_2,\,\cdot)^2\right).
\end{aligned}$$

However, there is no simple way to relate this to the standard coordinates on this *space. This behaviour is typical — while it is very easy to combine solutions of the Joyce equations, it is not generally possible to find holomorphic coordinates on the resulting spaces explicitly.*

One nice consequence of this form, however, is that we can see this metric is conformally flat by considering the hyperbolic isometry

$$\tilde{\rho} = \frac{2\rho}{\rho^2+(\eta+1)^2} \qquad \tilde{\eta} = \frac{\rho^2+\eta^2-1}{\rho^2+(\eta+1)^2}.$$

This converts the boundary data for $S^2 \times \mathcal{H}^2$ into that of $\mathbb{C}^2$. Since this affects only the conformal factor of the Joyce metric, it follows that g_J is conformal to g_{flat},

$$g_J = \lambda g_{flat}$$

where λ is half the ratio of this conformal factor with that of the previous example, in these new coordinates:

$$\begin{aligned}\lambda &= \frac{\left(\frac{\tilde{\rho}^2}{\sqrt{\tilde{\rho}^2+\tilde{\eta}^2}}\right)}{2\left(\frac{\rho^2}{\sqrt{\rho^2+(\eta-1)^2}\sqrt{\rho^2+(\eta+1)^2}}\right)} \\ &= \frac{2\sqrt{\rho^2+(\eta-1)^2}}{\sqrt{\rho^2+(\eta+1)^2}\sqrt{4\rho^2+(\rho^2+\eta^2-1)^2}}.\end{aligned}$$

We have seen a particular choice of conformal factor for which the Joyce metric is not just self-dual but also scalar flat Kähler. This conformal factor satisfies the asymptotic conditions and hence extends to a metric on the degenerate orbits. By applying hyperbolic isometries to this metric, we can obtain a family of scalar flat Kähler metrics which extend to all but one orbit in M, the image of the point at infinity.

5.2 Self-dual Einstein metrics

We explore a choice of conformal factor given by [4] which, subject to a condition relating the two components of a given solution ϕ, gives us an Einstein metric. We show which sums of basic solutions admit such a metric, and that the asymptotic conditions (4.13) are satisfied for this conformal factor on an open set in $\bar{\mathcal{H}}^2$. We investigate the geometry of this open set and see that it is made up of a single disc and intersects the boundary precisely where a certain potential is positive.

We sketch the construction of the Einstein metrics. This result is due to Calderbank-Pedersen [4], and we quote it without proof.

Theorem 5.2.1. *[4] Let $F : \mathcal{H}^2 \to \mathbb{R}$ be an eigenfunction of the hyperbolic Laplacian with eigenvalue $\frac{3}{4}$,*

$$\frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial \eta^2} = \frac{3}{4}\frac{F}{\rho^2}.$$

Then setting $f = \rho^{\frac{1}{2}} F$,

$$\left.\begin{aligned} \phi_1 &= (f_\rho,\ \eta f_\rho - \rho f_\eta) \\ \phi_2 &= (f_\eta,\ \rho f_\rho + \eta f_\eta - f) \end{aligned}\right\} \tag{5.3}$$

is a Joyce solution and putting $\Omega^2 = \frac{\phi_1 \wedge \phi_2}{F^2}$, the Joyce metric

$$g = \frac{\phi_1 \wedge \phi_2}{F^2}\left(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\det(\phi_1,\, \cdot)^2 + \det(\phi_2,\, \cdot)^2}{(\phi_1 \wedge \phi_2)^2}\right)$$

is Einstein on the region $F > 0$.

In particular if (ϕ_1, ϕ_2) is also a sum of basic solutions with convex non-singular boundary data then there is an Einstein metric conformal to the scalar flat Kähler metric. It is then possible to show the same asymptotic conditions hold for the Einstein metric, following [5].

We first examine the restrictions that this condition places on the sums of basic solutions and show that the asymptotic conditions for the Einstein conformal factor are equivalent to the conditions on the Kähler factor.

Proposition 5.2.2. *[5] If*

$$\phi = \sum f^{(y_i)} \otimes \underline{u}_i + \underline{u}_\infty,$$

a finite sum of basic solutions satisfying (5.3), then there are some λ_i, λ_∞ for which

$$\underline{u}_\infty = (0,\ \lambda_\infty) \text{ and } \underline{u}_i = (\lambda_i,\ y_i\lambda_i), \qquad \forall i \le n.$$

Proof. Consider a basic solution,

$$\begin{aligned}
\phi_1 &= \frac{\rho}{\sqrt{\rho^2 + (\eta - y)^2}} \otimes (1,\, \alpha) \\
\phi_2 &= \frac{\eta - y}{\sqrt{\rho^2 + (\eta - y)^2}} \otimes (1,\, \alpha).
\end{aligned}$$

If this satisfies (5.3) then

$$f = \sqrt{\rho^2 + (\eta - y)^2} + \beta,$$

and

$$\begin{aligned}
\eta f_\rho - \rho f_\eta &= \frac{y\rho}{\sqrt{\rho^2 + (\eta - y)^2}} = y f_\rho \\
\rho f_\rho + \eta f_\eta - f &= \frac{y(\eta - y)}{\sqrt{\rho^2 + (\eta - y)^2}} - \beta = y f_\eta - \beta.
\end{aligned}$$

Then we must have $\beta = 0$ and $\alpha = y$. The only basic solution which is not a scalar multiple of such a ϕ is $\phi^{(\infty)}$, so any linear combination of basic solutions satisfying (5.2.1) must have the form

$$\phi = \sum f^{(y_i)} \otimes (\lambda_i,\, \lambda_i y_i) + (0,\, \lambda_\infty), \qquad \lambda_i,\, \lambda_\infty \in \mathbb{R}.$$

□

On the other hand we can see this as a condition on the position of the vertices — this condition will then be equivalent to asking that $\lim_{\rho \to 0} f$ be continuous, following [5].

Proposition 5.2.3. *[5] Taking a finite sum of basic solutions as in (5.2.2), let*

$$f_0(\eta) = \lim_{\rho \to 0} f(\rho,\, \eta).$$

Then f_0 is piecewise linear with

$$f_0(\eta) = m\eta - n$$

on an edge with boundary data $(m,\, n)$ and this function is continuous.

Proof. Take a point $(0,\, y)$ on the edge with boundary data $(m,\, n)$. since

$$\phi = \sum_{i=0}^{n} f^{(y_i)} \otimes \underline{u}_i + \underline{u}_\infty$$

the potential is

$$f(\rho,\, \eta) = \sum_{i=0}^{n} \sqrt{\rho^2 + (\eta - y_i)^2} \otimes u_{i\,1} + \lambda$$

where $\underline{u}_i = (u_{i\,1},\, u_{i\,2})$ and λ is a constant. Since $y \neq y_0, \ldots,\, y_n$, for sufficiently small ρ we can apply a binomial expansion to each of these terms close to $(0,\, y)$. Taking these together gives us a Taylor series

$$f(\rho,\, \eta) = a(\eta) + b(\eta)\rho + c(\eta)\rho^2 + O(\rho^3)$$

for $\eta \in (y - \epsilon,\, y + \epsilon)$, $\epsilon > 0$. Then

$$\begin{aligned} \lim_{\rho \to 0} f_\eta &= a'(\eta) = m \\ \lim_{\rho \to 0} f_\rho &= b(\eta) = 0 \end{aligned}$$

using the asymptotic values found in (4.2.2), so that $a(\eta) = m\eta + \lambda$. Then

$$\lim_{\rho \to 0} \rho f_\rho + \eta f_\eta - f = -\lambda = n.$$

Hence on $(y - \epsilon,\, y + \epsilon)$

$$f_0(\eta) = m\eta - n.$$

Now consider a vertex y between edges with data $(m,\, n)$ and $(m',\, n')$. Then by (5.2.2) the basic solution corresponding to this vertex,

$$\phi_0 = \left(\frac{\rho}{\sqrt{\rho^2 + (\eta - y_i)^2}} \otimes (\lambda_i,\, y_i\lambda_i),\; \frac{\eta - y_i}{\sqrt{\rho^2 + (\eta - y_i)^2}} \otimes (\lambda_i,\, y_i\lambda_i) \right) \tag{5.4}$$

must have

$$\begin{aligned} \lambda_i &= \frac{1}{2}(m' - m) \\ \lambda_i y_i &= \frac{1}{2}(n' - n) \end{aligned}$$

so that

$$y_i = \frac{n' - n}{m' - m}. \tag{5.5}$$

Hence

$$\lim_{\eta \to y^-} f_0(\eta) = my - n = m'y - n' = \lim_{\eta \to y^+} f_0(\eta)$$

so f_0 is continuous. $\square$

Theorem 5.2.4. *[5] Let ϕ be a sum of basic solutions of the form (5.3) with convex boundary data $\{(m_i, n_i)\}_{i=0}^n$ such that $m_i \geq 0 \ \forall i \leq n$ and the asymptotic conditions (4.13) hold for the Kähler conformal factor. Then the Einstein metric (5.2.1) extends to all boundary points $(0, y)$ with neighbourhoods $U \subseteq \bar{\mathcal{H}}^2$ for which*

$$f_0(y) = \lim_{\rho \to 0} f(\rho, y) > 0.$$

Proof. Compare the conformal factors of the scalar flat Kähler and Einstein metrics — the Kähler metric had

$$\Omega^2_{SFK} = \rho \phi_1 \wedge \phi_2$$

whereas the Einstein metric is given by

$$\Omega^2_{SDE} = \frac{\phi_1 \wedge \phi_2}{F^2}.$$

Since the ratio of these is

$$\frac{1}{\rho F^2} = \frac{1}{f^2}$$

and is bounded near any boundary point with $f_0(y) > 0$, the asymptotic conditions (4.2.5) hold for the second precisely when they hold for the first, and therefore the metric extends to the boundary orbits, $\{(0, y) | f_0(y) > 0\}$. $\square$

Before we can understand the spaces constructed in this way we will need to understand the topology of the region $\{f > 0\}$ on which the metric is defined. Our method here follows the approach in [5].

Proposition 5.2.5. *[5] Let ϕ be a sum of basic solutions satisfying (5.3) and (4.9), and that the boundary data $\underline{w}$ has, for sufficiently large y,*

$$\underline{w}(y) = (0, -1) = -\underline{w}(-y).$$

Let

$$D^+ = \{(\rho,\,\eta) \in \mathcal{H}^2 | f(\rho,\,\eta) > 0\} \qquad Z = \{(\rho,\,\eta) \in \mathcal{H}^2 | f(\rho,\,\eta) = 0\}.$$

Z is a curve, has one component, and limit points $(0,\,z)$ *and* $(0,\,\infty)$.

Proof. At a point $(\rho,\,\eta) \in Z$, since the boundary data is convex,

$$\begin{aligned} 0 < \phi_1 \wedge \phi_2 &= \det \begin{pmatrix} f_\rho & f_\eta \\ \eta f_\rho - \rho f_\eta & \rho f_\rho + \eta f_\eta - f \end{pmatrix} \\ &= \rho(f_\rho^2 + f_\eta^2). \end{aligned}$$

Then $df \neq 0$ and by the implicit function theorem $Z = \{f = 0\}$ is a smooth 1-manifold.

Now note that since f_0' is the first component of $\underline{w}$,

$$f(\rho,\,\eta) = \int \frac{\rho^2}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}} f_0(y)\,dy. \tag{5.6}$$

Then

$$\begin{aligned} \frac{\partial f}{\partial \eta} &= -3\rho^2 \int \frac{\eta - y}{(\rho^2 + (\eta - y)^2)^{\frac{5}{2}}} f_0(y)\,dy \\ &= \rho^2 \int \frac{f_0'(y)}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}}\,dy \end{aligned}$$

after integrating by parts. Since the boundary data is convex and $\underline{w}(y) = (0,\,-1)$ for large y, the first component of $\underline{w}$ is non-negative and f_0 is increasing. Since f_0 is increasing (and non-constant on at least one interval) this is positive. Therefore Z meets each contour $\{\eta = c\}$ in at most one point.

$f_0(y)$ cannot be constantly zero on an interval since (5.2.3) would require that the boundary data be $(0,\,0)$, so there is some z such that f_0 is positive for $y > z$ and negative for $y < z$. Then Z has a unique component with limit point $(0,\,z)$ and cannot have closed components in $\mathcal{H}^2$. The final case we must eliminate is that Z has a second component with both end points at ∞ and the two components never meeting the same contour $\{\eta = c\}$. We can eliminate this case since $\frac{\partial f}{\partial \eta} > 0$, so that D^+ must lie to the right of Z. Then Z must consist of a single component, and $(0,\,z)$, $(0,\,\infty)$ are its end points. $\square$

Since the metric depends only on F^2 we also get an Einstein metric on

$$D^{-} = \{(\rho,\, \eta)|F(\rho,\, \eta) < 0\} \cup \{(0,\, y)|f_0(y) < 0\}$$

in the same way.

Example 5.2.6. • $\mathbb{C}P^2$. *Consider the boundary data for $\mathbb{C}P^2$ (3.3.2 and 4.2.8). While this is not of the form (5.2.2), if we change basis and move the vertices we can obtain boundary data*

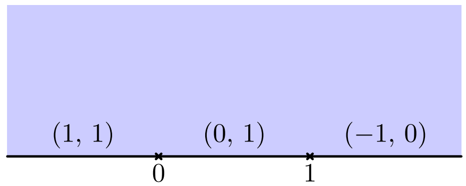


Figure 5.1: Boundary data for $\mathbb{C}P^2$.

While this boundary data is not convex, we instead have

$$\det(\underline{w}(y),\, \underline{w}(z)) \geq 0 \qquad \textit{almost all } y < z$$

so $\phi_1 \wedge \phi_2$ is everywhere negative and this then gives an Einstein metric as usual. In fact this change of sign means that the resulting Einstein metric has positive scalar curvature ([4]).

Then we have a Joyce solution

$$\begin{aligned}
\phi_1 &= \frac{\rho}{2\sqrt{\rho^2+\eta^2}}(-1,\, 0) + \frac{\rho}{2\sqrt{\rho^2+(\eta-1)^2}}(-1,\, -1) \\
\phi_2 &= \frac{\eta}{2\sqrt{\rho^2+\eta^2}}(-1,\, 0) + \frac{\eta-1}{2\sqrt{\rho^2+(\eta-1)^2}}(-1,\, -1) + \frac{1}{2}(0,\, 1)
\end{aligned}$$

which is now of the form (5.2.2). The corresponding potential is

$$f(\rho,\, \eta) = -\frac{\sqrt{\rho^2+\eta^2}}{2} - \frac{\sqrt{\rho^2+(\eta-1)^2}}{2} - \frac{1}{2}.$$

This gives us a self-dual Einstein metric of positive scalar curvature on $\mathbb{C}P^2$ [4], so by uniqueness this is the Fubini-Study metric.

- S^4. *Considering the boundary data for $\mathbb{C}^2$ seen in (4.2.8) we can again change basis and obtain boundary data*

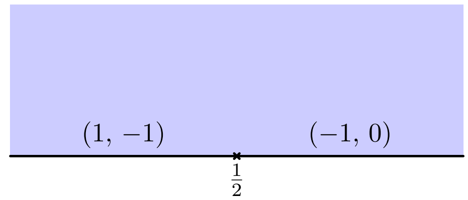


Figure 5.2: Boundary data for S^4.

with Joyce solution

$$\begin{aligned}\phi_1 &= \frac{\rho}{2\sqrt{\rho^2+(\eta-\frac{1}{2})^2}}(-2,\,1)\\ \phi_2 &= \frac{\eta-\frac{1}{2}}{2\sqrt{\rho^2+(\eta-\frac{1}{2})^2}}(-2,\,1)+\frac{1}{2}(0,\,-1)\end{aligned}$$

and potential

$$f(\rho,\,\eta)=-\frac{1}{2}-\sqrt{\rho^2+\left(\eta-\frac{1}{2}\right)}.$$

Now, we saw (5.1.3) that $g_{SFK}=2g_{flat}$ *and in (5.2.4) that*

$$g_{SDE}=\frac{1}{f^2}g_{SFK}.$$

Let $(\tilde{\rho},\,\tilde{\eta})$ *be coordinates on the base space of the scalar flat Kähler manifold and* $(r_1,\,\theta_1,\,r_2,\,\theta_2)$ *radial coordinates on* $\mathbb{C}^2$*. If we identify* $(\rho,\,\eta)=(\tilde{\rho},\,\tilde{\eta}+\frac{1}{2})$*, then*

$$f(\rho,\,\eta)=\tilde{\rho}^2+\left(\tilde{\eta}+\frac{1}{2}\right)^2=(r_1^2+r_2^2)^2$$

and

$$g_{SDE}=\frac{2}{(\frac{1}{2}+r_1^2+r_2^2)^2}(dr_1^2+r_1^2d\theta_1^2+dr_2+r_2^2d\theta_2).$$

Then if we set $\tilde{r}_1=\sqrt{2}r_1,\ \tilde{r}_2=\sqrt{2}r_2$ *then*

$$\begin{aligned}g_{SDE} &= \frac{2}{(1+\tilde{r_1}^2+\tilde{r_2}^2)^2}(d\tilde{r}_1+\tilde{r}_1^2d\theta_1^2+d\tilde{r}_2^2+\tilde{r}_2^2d\theta_2^2)\\ &= 2g_{S^4}\end{aligned}$$

therefore this is the spherical metric on S^4*.*

- $S^2\times\mathcal{H}^2$*. Consider the boundary data of* $S^2\times\mathcal{H}^2$ *(4.2.8). These is no Einstein* *metric on $S^2 \times \mathcal{H}^2$, but we can find a self-dual Einstein metric of the type seen in (5.2.5) over part of the space. In order to construct a self-dual Einstein metric we will need to move the vertices,*

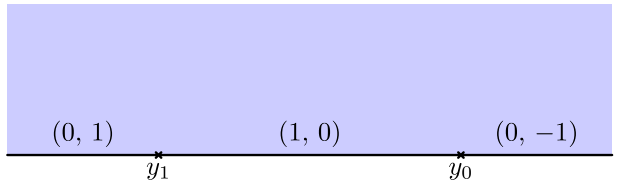


Figure 5.3: Boundary data for $S^2 \times \mathcal{H}^2$.

Then the corresponding boundary data is

$$\phi = \frac{f^{(y_0)}}{2}(-1,\,-1) + \frac{f^{(y_1)}}{2}(1,\,-1)$$

and by (5.2.2) this gives an Einstein metric precisely when $y_0 = 1,\, y_1 = -1$. This gives

$$f_0(y) = \begin{cases} 1 & y \geq 1 \\ y & -1 < y \leq 1 \\ -1 & y \leq -1 \end{cases}$$

and hence

$$f(y) = \frac{\eta - 1}{2\sqrt{\rho^2 + (\eta - 1)^2}} - \frac{\eta + 1}{2\sqrt{\rho^2 + (\eta + 1)^2}}.$$

This is then the potential of a self-dual Einstein metric on the non-compact region $\pi^{-1}(D^+)$, where

$$\begin{aligned} D^+ &= \{(\rho,\,\eta) \in \bar{\mathcal{H}}^2 | f(\rho,\,\eta) > 0\} \\ &= \{(\rho,\,\eta) \in \bar{\mathcal{H}}^2 | \eta > 0\}. \end{aligned}$$

5.3 Einstein boundary data from continued fractions

We have seen that, given boundary data satisfying a linear condition (5.3), the Joyce conformal metric has an Einstein representative. However, there is another way to encode this data using continued fractions, as used in [5].

Let $\frac{p}{q} = \alpha \in (0,\, 1) \cap \mathbb{Q}$. As we saw in (3.1.21) there is a unique sequence $e_1,\, \ldots,\, e_k \in \mathbb{Z}$ with

$$\alpha = \cfrac{1}{e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ldots - \cfrac{1}{e_k}}}}$$

and defining truncated terms

$$\frac{n_{j+1}}{m_{j+1}} = \cfrac{1}{e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ldots - \frac{1}{e_j}}}}$$

gives us a sequence of vectors $(m_0,\, n_0) = (0,\, -1),\, (1,\, 0),\, \ldots,\, (m_k,\, n_k),\, (m_{k+1},\, n_{k+1}) = (p,\, q)$ with

$$\det \begin{pmatrix} m_j & m_{j+1} \\ n_j & n_{j+1} \end{pmatrix} = 1 \qquad\qquad 0 \leq j \leq k.$$

Now suppose that $e_j \geq 3\ \forall j$, and let

$$y_j = \frac{n_{j+1} - n_j}{m_{j+1} - m_j}.$$

Lemma 5.3.1. *[5] With $(m_j,\, n_j)$ and y_j defined as above, if $e_j \geq 3\ \forall j$ then the y_j form a strictly decreasing sequence.*

Proof. For each $j \geq 1$ we have (from (3.1.21))

$$\begin{aligned} n_j m_{j+1} - m_j n_{j+1} &= -1 \\ n_{j-1} m_{j+1} - m_{j-1} n_{j+1} &= -e_j. \end{aligned}$$

Solving these gives

$$(m_{j+1},\, n_{j+1}) = e_j(m_j,\, n_j) - (m_{j-1},\, n_{j-1}).$$

In particular since $m_0 = 0$ and $m_1 = 1$, the m_j form a strictly increasing sequence.

Now, calculating the difference between the y_j and using the two determinants again gives

$$y_{j-1} - y_j = \frac{e_j - 2}{(m_{j+1} - m_j)(m_j - m_{j-1})}.$$

Since the denominator is positive and $e_j \geq 3$ this is strictly positive and the the y_j are decreasing. $\square$

Then we can define a boundary data function

$$\underline{w}(y) = \begin{cases} (0,\, -1) & 1 < y \\ (1,\, 0) & y_0 < y \leq 1 \\ (m_j,\, n_j) & y_j < y \leq y_{j-1},\, \forall j \geq 1 \\ (0,\, 1) & y \leq \alpha \end{cases}.$$

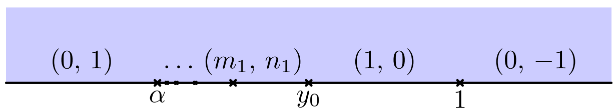


Figure 5.4: The boundary data function.

Since the vertices satisfy (5.5), this boundary data function satisfies the Einstein condition (5.3). In fact the space given by this boundary data corresponds to a resolution of an orbifold singularity

$$\frac{\mathbb{C}^2}{\begin{pmatrix} \omega & \\ & \omega^q \end{pmatrix}},$$

where ω is a p-th root of unity, and this is why the continued fraction expansion of (3.1.21) appears.

5.4 Smeared solutions

A further generalisation is given by Calderbank-Singer in [5], which replaces the step functions we have used for our boundary data with a distribution. We refer to such a solution as a *smeared solution*. The metric will now be defined only over a non-compact region, since we will only be able to extend the metric to the degenerate orbits where the boundary data restricts to a step function.

The proof of this proceeds by first showing that the asymptotic conditions (4.13) are satisfied on a neighbourhood of all suitable boundary points then building an open set on which a conformal metric is defined from such points.

Let $\underline{u}$ be an $\mathbb{R}^2$-valued, compactly supported distribution. That is, for any smooth compactly supported function $\psi : \mathbb{R} \to \mathbb{R}$,

$$\left| \int \psi(y)\underline{u}(y)\, dy \right| < \infty,$$

and there is some compact $K \subseteq \mathbb{R}$ such that if $\operatorname{Supp}\psi \cap K = \emptyset$,

$$\int \psi(y)\underline{u}(y)\, dy = 0.$$

Since $\underline{u}$ is compactly supported, the Joyce solution

$$\begin{aligned} \phi_1 &= \int \frac{\rho}{\sqrt{\rho^2 + (\eta - y)^2}} \underline{u}(y)\, dy \\ \phi_2 &= \int \frac{\eta - y}{\sqrt{\rho^2 + (\eta - y)^2}} \underline{u}(y)\, dy + \underline{u}_\infty \end{aligned}$$

exists, even though the functions we convolve with are not themselves compactly supported. Then, as in (4.6), we can integrate by parts to find the boundary data as a distribution,

$$\begin{aligned} \phi_1 &= \int \frac{\rho(y - \eta)}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}} \underline{w}(y)\, dy \\ \phi_2 &= \int \frac{\rho^2}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}} \underline{w}(y)\, dy. \end{aligned}$$

We will abuse notation by treating $\underline{w}$ as a (locally constant) function where $\underline{u} = 0$.

The following pair of results then follow from the proofs of (4.2.2) and (4.2.5):

Proposition 5.4.1. *[5] If ϕ is a smeared solution with boundary data $\underline{w}$, a $y \in \mathbb{R}$ for which there is some $\delta > 0$ with*

$$\underline{w}(y) = \begin{cases} (m, n) & z - \delta < y < z \\ (m', n') & z < y < z + \delta \end{cases}$$

and

$$\det\begin{pmatrix} m & m' \\ n & n' \end{pmatrix} = -1$$

then on some (possibly smaller) interval $(y-\delta_2,\, y+\delta_2) \subseteq \bar{\mathcal{H}}^2$, the asymptotic conditions (4.11) and (4.12) are satisfied.

Proposition 5.4.2. *[5] If ϕ is a smeared solution with boundary data $\underline{w}$ and $\underline{w}$ is locally constant at y, with $\underline{w}(y) = (m,\, n)$ a primitive vector and $\underline{w}$ is convex then there is a neighbourhood of $(0,\, y) \in \bar{\mathcal{H}}^2$ on which the asymptotic conditions (4.13) are satisfied.*

Then putting these results together,

Theorem 5.4.3. *[5] If*

$$(\phi_1,\, \phi_2) = \left(\int \frac{\rho(y-\eta)}{(\rho^2+(\eta-y)^2)^{\frac{3}{2}}}\underline{w}(y)dy,\, \int \frac{\rho^2}{(\rho^2+(\eta-z)^2)^{\frac{3}{2}}}\underline{w}(z)dz\right)$$

with $\underline{w}$ a distribution, and $V \subseteq \partial\mathcal{H}^2 \setminus \{(0,\, \infty)\}$ is an open subset such that for each $(0,\, y) \in V$, either

1. *$\underline{w}(y) = (m,\, n)$ is locally constant, and $(m,\, n)$ is primitive, or*

2. $$\underline{w}(z) = \begin{cases} (m',\, n') & y > z \\ (m,\, n) & y < z \end{cases}$$

   on some neighbourhood of $(0,\, y)$, with $\det\begin{pmatrix} m & m' \\ n & n' \end{pmatrix} = -1$

and if $\underline{w}$ is convex, then the Joyce metric generated by ϕ extends to $\mathcal{H}^2 \cup V$.

In particular if we take a sum of basic solutions with convex boundary data, we can perturb the metric by adding a smooth function on a compact set. Then, providing convexity is preserved, the resulting solution induces a metric away from the support of the perturbing function.

5.5 Joyce's non-simply connected self-dual spaces

In ([23], 3.4.2), Joyce constructs a family of self-dual toric manifolds with non-trivial fundamental group by taking a quotient of one of the metrics constructed above by a group action. This is achieved by taking a matrix $R \in SL(2, \mathbb{Z})$ with eigenvalues r, r^{-1} and eigenvectors X, Y, such that $r > 1$ and vectors $v_1, \ldots, v_k$ and $w_1, \ldots, w_l \in \mathbb{Z}^2$ primitive vectors with

$$\begin{aligned} \det(v_j, v_{j+1}) &= \det(w_j, w_{j+1}) = -1 \\ \det(Rv_k, v_1) &= \det(w_l, Rw_1) = -1 \end{aligned}$$

and

$$\begin{array}{ll} \det(X, Y) < 0 & \\ \det(v_j, X) > 0 & \det(v_j, Y) < 0 \\ \det(w_j, Y) > 0 & \det(w_j, X) > 0 \end{array}$$

and sequences of points $p_1, \ldots, p_k$ and $q_1, \ldots, q_l \in \mathbb{R}$ and $v > 1$ such that

$$p_1 < p_2 < \ldots < p_k < v^{-1}p_1 < 0 < q_1 < \ldots < q_l < vq_1.$$

We then have boundary data

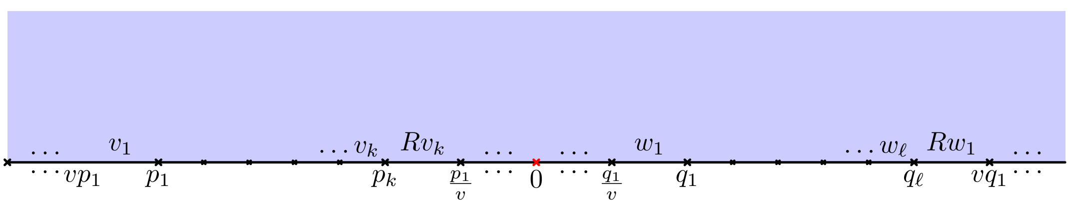


Figure 5.5: Joyce's boundary data

The corresponding Joyce solution is an infinite series, and the conditions $v > 1$ and $r > 1$ ensure this series converges, and by construction it is invariant under the $\mathbb{Z}$-action

$$n \cdot \left(\rho, \eta, \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}\right) \mapsto \left(v^n\rho, v^n\eta, R^n \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}\right). \tag{5.7}$$

However, it is not immediately clear what the toric manifold over which this should give us a Joyce metric is — we cannot use symplectic reduction as in (3.2.3) here, since the orbit space now has infinitely many edges. Instead we can construct the toric manifold locally:

Let $B = \bar{\mathcal{H}}^2 \setminus \{(0, 0), (0, \infty)\}$, $\underline{w}$ the boundary data function as above, and $\tilde{M}$ the torus fibration over B with this boundary data. That is, take $B \times T^2$ and over each boundary point $(0, y) \in B$ collapse the circle orthogonal to $\underline{w}(y)$ if $\underline{w}$ is locally constant at y, or the entire torus if $\underline{w}$ is not locally constant.

To show this topological space is a smooth manifold, note that for any open set $U \subseteq B$ containing only finitely many vertices we can view $\underline{w}$ as a step function on $U \cap \partial\mathcal{H}^2$ perturbed by a distribution supported outside U,

$$\underline{w} = \underline{w}|_U + (\underline{w} - \underline{w}|_U)$$

and hence by (5.7) $\pi^{-1}(U)$ is a smooth manifold. Then let

$$\begin{aligned} U_i &= (v^{-i}q_1,\, v^{i+1}q_1) \cup \mathcal{H}^2 \\ V_i &= (v^i p_1,\, v^{-i-1}p_1) \cup \mathcal{H}^2 \end{aligned} \qquad i \geq 0.$$

Each of these contains only finitely many vertices, and it is clear that the transition maps between U_i and V_i are smooth on their intersection. This provides a smooth atlas for B. Then this Joyce solution gives us a self-dual metric on the quotient of $\tilde{M}$ by the $\mathbb{Z}$-action. This space is a compact 4-manifold, and has fundamental group $\pi_1 \cong \mathbb{Z}$.

5.6 Einstein manifolds from infinite continued fractions

In [6] an extension to the method of section (5.3) is given, in which the rational $\frac{p}{q} \in (0,\, 1) \cap \mathbb{Q}$ is replaced with an irrational $\alpha \in (0,\, 1)$. This replaces the finite continued fraction with an infinite continued fraction,

$$\alpha = \cfrac{1}{e_1 - \cfrac{1}{e_2 - \cfrac{1}{\cdots}}}$$

and now gives us an infinite sequence of vectors $(m_j,\, n_j)$. Whereas in the finite case Calderbank-Singer extended this boundary data by a constant function $(0,\, 1)$, they here introduce an *odd extension* of the potential f_0,

$$f_0(y) = \begin{cases} 1 & y > 1 \\ y & y_0 < y \leq 1 \\ m_j y - n_j & y_j < y \leq y_{j+1}\, \forall j \geq 2 \\ -f_0(\alpha - y) & y < \alpha \end{cases} \tag{5.8}$$

to extend the boundary data. The symmetry of this potential simplifies the region D^+ over which the manifold is defined. In chapter 6 we show that using a different extension weakens the conditions required to prove completeness of these metrics.

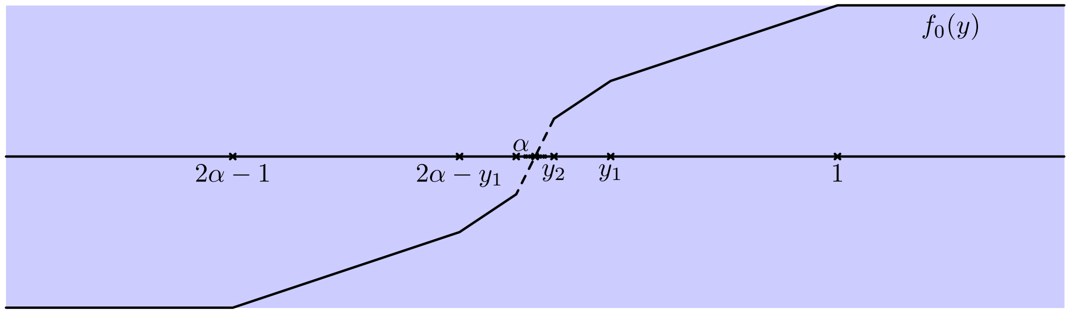


Figure 5.6: The odd extension.

Calderbank-Singer show that, provided for some $N \in \mathbb{N}$

$$3 \leq e_j \leq N \qquad\qquad \forall j \geq 1$$

this boundary data is convex and gives a well-defined Joyce solution, and the corresponding self-dual Einstein metric is complete. The proof of this result rests on two bounds — firstly an upper bound of f and secondly a lower bound on $\phi_1 \wedge \phi_2$ close to the singularity. They then consider curves approaching the boundary of the space and show that any such curve must have infinite length, and hence conclude that the space is complete. In chapter 6 we will give a generalisation of this result showing that the assumption that the sequence $(e_j)_{j=1}^{\infty}$ be bounded above is not necessary. Since this result will make use of several of the results found in this paper, we give them here in some detail.

In order to bound f we will first need to show that the sequence $(m_j)_{j=1}^{\infty}$ grows sufficiently quickly.

Lemma 5.6.1. *[6] With the sequence $(m_j)_{j=1}^{\infty}$ defined by an infinite continued fraction as above, if $e_j \geq 3$ $\forall j \geq 1$ then*

$$\frac{m_{j+1}}{m_j} > \tau^2 \quad\quad j \geq 0$$

where τ is the golden ratio,

$$\tau = \frac{1+\sqrt{5}}{2}.$$

Proof. Clearly this holds for $j = 0$. We use (3.1.21) and $e_j \geq 3$ to proceed inductively — if $m_j > \tau^2 m_{j-1}$, then

$$\begin{aligned} m_{j+1} &= e_j m_j - m_{j-1} \\ &> 3m_j - \tau^{-2} m_j = \tau^2 m_j. \end{aligned}$$

□

Using this bound, Calderbank-Singer then find a bound for f_0:

Proposition 5.6.2. *[6] With f_0 defined as above, for some $C > 0$*

$$f_0(y) < C\sqrt{y-\alpha}.$$

Proof. We saw in the previous lemma that $(m_j)_{j=0}^{\infty}$ forms an increasing sequence, so $f_0|_{(\alpha,\infty)}$ is concave. Then it is sufficient to prove this bound holds at the vertices y_j.

Central to this calculation will be two bounds on $y_{j-1} - y_j$.

$$\begin{aligned} y_{j-1} - y_j &= \frac{e_j - 2}{(m_j - m_{j-1})(m_{j+1} - m_j)} \\ &= \frac{e_j - 2}{\left(1 - \frac{m_{j-1}}{m_j}\right)\left(e_j - 1 - \frac{m_{j-1}}{m_j}\right)} \frac{1}{m_j^2}. \end{aligned}$$

Since $\frac{m_{j-1}}{m_j} < \tau^2 < \frac{1}{2}$,

$$y_{j-1} - y_j < \frac{2}{m_j^2}.$$

Then

$$\begin{aligned} y_n - \alpha &> \sum_{j=n+1}^{\infty} y_{j-1} - y_j \\ &> \sum_{j=n+1}^{\infty} \frac{1}{2m_j^2} > \frac{1}{2m_{n+1}^2}. \end{aligned}$$

However, since $(f_0(y_n)) \to 0$, we also have

$$\begin{aligned} f_0(y_n) &= \sum_{j=n+1}^{\infty} f_0(y_{j-1}) - f_0(y_j) \\ &= \sum_{j=n+1}^{\infty} m_j(y_{j-1} - y_j) < \sum_{j=n+1}^{\infty} \frac{2}{m_j}. \end{aligned}$$

Then using the exponential bound on m_j from the previous lemma,

$$f_0(y_n) < \frac{2}{m_{n+1}} \sum_{j=0}^{\infty} \tau^{-2j} < \frac{4}{m_{n+1}}.$$

Then comparing these two estimates,

$$f_0(y_n) < 4\sqrt{2}\sqrt{y_n - \alpha}.$$

□

Then let

$$f(\rho, \eta) = \int \frac{\rho^2 f_0(y)}{\sqrt{\rho^2 + (\eta - y)^2}} \, dy$$

and $F = \rho^{-\frac{1}{2}} f$ as usual. Calderbank-Singer find estimates for this integral to bound f as follows:

Proposition 5.6.3. *[6] With f and f_0 as above, for some $D > 0$*

$$f_0(\rho, \eta) \leq D\sqrt{r}.$$

where $r = \sqrt{\rho^2 + (\eta - \alpha)^2}$, the polar coordinate around the singularity.

Proof. Since f_0 is negative on $(-\infty, \alpha)$ and is bounded by $C\sqrt{y - \alpha}$ on (α, ∞),

$$\begin{aligned} f(\rho, \eta) &= \int \frac{\rho^2}{\sqrt{\rho^2 + (\eta - y)^2}^3} f_0(y) \, dy \\ &\leq C \int_{\alpha}^{\infty} \frac{\rho^2}{\sqrt{\rho^2 + (\eta - y)^2}^3} \sqrt{y - \alpha} \, dy. \end{aligned}$$

Now we consider two regions close to $(0, \alpha)$. First consider the case $\eta > \alpha$, and set $\tilde{\eta} = \eta - \alpha$. Take coordinates $\theta = \frac{\rho}{\tilde{\eta}}$ and change variables to $z = \frac{y-\alpha}{\tilde{\eta}}$.

Then

$$f(\theta\tilde{\eta},\, \tilde{\eta}) \leq C\sqrt{\tilde{\eta}} \int_0^\infty \frac{\theta^2}{(\theta^2 + (z-1)^2)^{\frac{3}{2}}} \sqrt{z}\, dz$$

and by a further change of variables, $w = \frac{z-1}{\theta}$ we find

$$f(\theta\tilde{\eta},\, \tilde{\eta}) \leq C\sqrt{\tilde{\eta}} \int_0^\infty \frac{1}{(1+w^2)^{\frac{3}{2}}} \sqrt{1+\theta w}\, dw.$$

This is increasing in θ, so if we set

$$D_1 = \frac{C}{2} \int_0^\infty \frac{\sqrt{1+w}}{(1+w^2)^{\frac{3}{2}}}\, dw.$$

then

$$f(\theta\tilde{\eta},\, \tilde{\eta}) \leq f(\tilde{\eta}, \tilde{\eta}) \leq D_1\sqrt{\tilde{\eta}} \quad \forall 0 \leq \theta \leq 1.$$

For the second region suppose $\rho \neq 0$ and change coordinates to $\xi = \frac{\eta-\alpha}{\rho}$ and $x = \frac{y-\alpha}{\rho}$,

$$\begin{aligned} f(\rho,\, \xi\rho) &\leq C\sqrt{\rho} \int_0^\infty \frac{1}{(1+(\xi-x)^2)^{\frac{3}{2}}} \sqrt{x}\, dx \\ &= C\sqrt{\rho} \int_{-\phi}^\infty \frac{1}{(1+w^2)^{\frac{3}{2}}} \sqrt{w+\xi}\, dw \end{aligned}$$

where we have substituted $w = x - \xi$. This integral is uniformly bounded for $\phi < \lambda$ for any λ (note that here we allow $\xi < 0$), so there is a D_2 such that

$$f(\rho,\, \xi\rho) \leq D_2\sqrt{\rho} \leq D_2\sqrt{r} \quad \forall \xi < \lambda.$$

These two estimates together cover all of $\bar{\mathcal{H}}^2 \setminus \{(0,\, \alpha)\}$, so the result holds with $D = \max\{D_1,\, D_2\}$. □

Note that all we use about $f_0|_{(-\infty,\,\alpha)}$ in this proof is that it is non-positive. We will use this fact to substitute other potentials in chapter 6.

Since the Joyce solution with this potential is given by an infinite sum, we must check that it converges. As in section 5.4, it suffices to show that $\frac{dw}{dy}$ is a compactly supported distribution.

Lemma 5.6.4. *The potential (5.8) above is given by a well-defined Joyce solution.*

Proof. It suffices to check that the step function $\underline{w}$ is integrable. We will check this on $[\alpha, y_0]$, by symmetry the same then holds on $[2\alpha - y_0, \alpha]$, and these sets together form the support of $\frac{dw}{dy}$. It follows that $\frac{dw}{dy}$ is a compactly supported distribution, so the solution converges. We consider the first component first:

$$\int |w_1(y)|\, dy = \sum_{j=1}^{\infty}(y_{j-1} - y_j)m_j$$

but by (5.6.2), $(y_{j-1} - y_j) < \frac{2}{m_j^2}$ and by (5.6.1) $\frac{m_{j+1}}{m_j} > \tau^2$, so

$$\begin{aligned} \int |w_1(y)|\, dy &< \sum_{j=1}^{\infty} \frac{2}{m_j} \\ &< \sum_{j=1}^{\infty} \frac{2}{m_1}\tau^{2-2j} < \infty. \end{aligned}$$

Since $n_j < m_j \, \forall j \geq 1$ (this is easily checked from (3.1.21)) the same holds for the second component. $\square$

Then before concluding this solution yields an Einstein metric it just remains to prove this boundary data is convex.

Lemma 5.6.5. *The boundary data given by (5.8),*

$$\underline{w}(y) = \begin{cases} (0,\, -1) & y > 1 \\ (m_j,\, n_j) & y_j < y < y_{j-1} \\ (m_j,\, 2\alpha m_j - n_j) & 2\alpha m_j - y_{j-1} < y < 2\alpha - y_j \\ (0,\, 1) & y < 2\alpha - 1 \end{cases},$$

is convex.

Proof. Denote $\underline{w}_j = (m_j, n_j)$, $\underline{w}_{-j} = (m_j, 2\alpha m_j - n_j)$. As in section (5.3), it follows from (3.1.21) that

$$\det(\underline{w}_k, \underline{w}_j) \leq 0 \qquad \forall j \leq k$$

and for any $j < k$,

$$\det(\underline{w}_{-j}, \underline{w}_{-k}) = n_j m_k - n_k m_j = -\det(\underline{w}_j, \underline{w}_k) \leq 0.$$

Finally, for any $j,\, k \geq 1$,

$$\begin{aligned}\det(\underline{w}_{-j},\, \underline{w}_k) &= m_j n_k + m_k n_j - 2\alpha m_j m_k \\ &= m_j m_k \left(\frac{n_k}{m_k} - \alpha \right) + m_j m_k \left(\frac{n_j}{m_j} - \alpha \right).\end{aligned}$$

Now, $\left(\frac{n_j}{m_j}\right)_{j=1}^{\infty}$ is an increasing sequence, since

$$\frac{n_{j+1}}{m_{j+1}} - \frac{n_j}{m_j} = \frac{1}{m_j m_{j+1}} > 0,$$

and it converges to α. Then both of these terms are negative, and

$$\det(\underline{w}_{-j},\, \underline{w}_k) \leq 0 \qquad \forall j,\, k \geq 1.$$

□

Therefore this potential gives a well-defined Joyce solution and Einstein manifold.

Now, because f_0 is symmetric about α,

$$Z = \{(\rho,\, \eta) \in \mathcal{H}^2 | f(\rho,\, \eta) = 0\} = \{(\rho,\, 0) \in \mathcal{H}^2\}.$$

Calderbank-Singer then consider smooth curves approaching the boundary of

$$D^+ = \{(\rho,\, \eta) \in \mathcal{H}^2 | f(\rho,\, \eta) > 0\},$$

and by showing that any such curve has infinite length in the Einstein metric, are able to conclude that the space is complete. There are three types of boundary points — points in Z, $(0,\, \infty)$ and $(0,\, \alpha)$.

In order to show such curves are infinite we introduce a partial order on quadratic forms,

$$A \leq B \text{ if } B - A \text{ is positive semi-definite.} \tag{5.9}$$

Then in particular, if A and B are metrics, the length of a curve with respect to B is bounded by its length with respect to A.

Proposition 5.6.6. *([6], [5]) Let $\gamma : [0,\, 1) \to D^+$ be a curve with*

$$\lim_{t \to 1} \gamma(t) = (\rho_0,\, \eta_0) \in Z.$$

Then

$$\text{length}\,\gamma = \int_0^1 \sqrt{g(\gamma'(t),\,\gamma'(t))}\,dt = \infty.$$

Proof. This is implied by the analysis of Z for smeared solutions in [5], and is proved as follows: Take $(\rho_0,\,\eta_0) \in Z$. We have seen (5.2.5) that $df(\rho_0,\,\eta_0) \neq 0$, so by the implicit function theorem there is a neighbourhood $U \subseteq \mathcal{H}^2$ of $(\rho_0,\,\eta_0)$ and a diffeomorphism

$$\phi : U \to V \subseteq \mathbb{R}^2$$

such that $\phi(\rho_0,\,\eta_0) = (0,\,0)$ and $f \circ \phi^{-1}(x,\,y) = x$. Since ϕ is a diffeomorphism there exists $\lambda > 0$ and a (possibly smaller) neighbourhood $\tilde{U} \subseteq U$ such that

$$(d\rho^2 + d\eta^2)|_{\tilde{U}} \geq \lambda(dx^2 + dy^2)|_{\tilde{U}}.$$

By continuity we can restrict $\tilde{U}$ further so that $\rho < 2\rho_0$ and

$$\phi_1 \wedge \phi_2(\rho,\,\eta) > \frac{1}{2}\phi_1 \wedge \phi_2(\rho_0,\,\eta_0) > 0.$$

Then putting $F = \rho^{\frac{1}{2}} f$ as usual,

$$\begin{aligned} g_{\underline{w}}\big|_{\tilde{U}} &\geq \left.\frac{\phi_1 \wedge \phi_2}{F^2}\,\frac{d\rho^2 + d\eta^2}{\rho^2}\right|_{\tilde{U}} \\ &> \left.\frac{\phi_1 \wedge \phi_2(\rho_0,\,\eta_0)}{2}\,\frac{1}{f^2}\,\lambda\frac{dx^2 + dy^2}{2\rho_0}\right|_{\tilde{U}} \\ &= \left.\lambda\frac{\phi_1 \wedge \phi_2(\rho_0,\,\eta_0)}{4\rho_0}\,\frac{dx^2 + dy^2}{f^2}\right|_{\tilde{U}} \end{aligned}$$

and since $x = f(\rho,\,\eta)$, this is just a multiple of the hyperbolic metric with boundary at Z. Hence γ has infinite length. $\square$

For curves approaching $(0,\,\infty)$ we observe that the behaviour of the metric near this point is controlled by the values of $f_0(y)$ for large $|y|$:

Proposition 5.6.7. *[6] Let $\gamma : [0,\,1) \to \pi^{-1}(D^+)$ be a curve with*

$$\lim_{t \to 1} \pi(\gamma(t)) = (0,\,\infty).$$

Then

$$\text{length}\,\gamma = \int_0^1 \sqrt{g(\gamma'(t),\,\gamma'(t))}\,dt = \infty.$$

Proof. It follows from (4.2.1) that applying an inversion to $\mathcal{H}^2$ transforms the boundary data so that

$$\tilde{f}_0(\tilde{y}) = |\tilde{y}| f_0\left(-\frac{1}{\tilde{y}}\right).$$

In particular, close to (0, 0) the transformed potential is

$$\tilde{f}_0(\tilde{y}) = \tilde{y}$$

so that $\tilde{f}$ vanishes to first order near this point. Since the boundary data is convex, however, the determinant $\tilde{\phi}_1' \wedge \tilde{\phi}_2$ is positive near this point. So for some $C > 0$ and some neighbourhood $\tilde{U}$ of (0, 0),

$$\begin{aligned} g(u,\, u)|_{\tilde{U}} &\geq \tilde{\phi}_1' \wedge \tilde{\phi}_2 \frac{d\tilde{\rho}^2 + d\tilde{\eta}^2}{\tilde{f}^2}\bigg|_{\tilde{U}} \\ &\geq C\frac{d\tilde{\rho}^2 + d\tilde{\eta}^2}{\tilde{f}^2}\bigg|_{\tilde{U}} \end{aligned}$$

where the inequality is in the sense of (5.9). Since f vanishes at (0, 0), the curve γ has length ∞. □

Finally Calderbank-Singer consider curves approaching $(0,\, \alpha)$. By imposing the condition that for some $N \in \mathbb{N}$,

$$e_j \leq N \qquad\qquad \forall j \geq 1$$

they are able to prove that $\phi_1' \wedge \phi_2$ is sufficiently large close to the singularity that, given (5.6.3), such a curve is infinite in length.

Chapter 6

Spaces of infinite topological type

We consider metrics constructed by taking infinite sums of basic solutions. By allowing the vertices to converge to a point we create a singularity, and we investigate the behaviour of Joyce metrics near such a point.

We show that, if the convergence is not too rapid, for one choice of conformal factor we obtain a complete Kähler metric. The construction of these metrics is similar to that of the non-simply connected metrics found by Joyce ([23], cf. section 5.5), but the completeness of the metrics is new. The family of metrics constructed in this way includes the Ricci-flat metrics of Andersen-Kronheimer-LeBrun [26].

We also consider Einstein metrics obtained from infinite sums of basic solutions, and, by using estimates found in the previous section 5.6, extend the construction of [6] to produce new complete self-dual Einstein metrics, including metrics on spaces containing chains of spheres whose self-intersection is not bounded.

6.1 The scalar flat Kähler metric

In chapter 5 we saw a family of conformal scalar flat Kähler metrics for each convex non-singular boundary data function by taking finite sums of basic solutions. We now replace this finite sum with a series of basic solutions with a single limit point α.

The family of metrics we find in this way includes the Ricci-flat Kähler manifolds of infinite topological type found by Anderson, Kronheimer and LeBrun in [26], found by applying the Gibbons-Hawking ansatz to a potential

$$V(p) = \frac{1}{2}\sum_{i=0}^{\infty} \frac{1}{\|p - p_i\|}$$

with the singular points p_i chosen so as to make the sum converge. In our picture these metrics will correspond to fixing the boundary data on each interval and varying the position of the vertices. We see these metrics more explicitly at the end of the section.

We begin by choosing a boundary data function — let $\underline{u}$ be a compactly supported distribution with $\frac{dw}{dy} = \underline{u}$. This condition guarantees the solution will converge. Now let $\underline{w}$ be locally constant except at a countable collection of points $(y_j)_{j=0}^{\infty}$. We can think of this as a step function with infinitely many steps. We assume first that these points form a decreasing sequence with $\lim_{j\to\infty} y_j = \alpha$, that is, that $\alpha < \ldots < y_1 < y_0$, as in diagram (6.1), and generalise this to other countable collections with a single limit point. In particular we end the section by allowing that the collection of vertices contains subsequences approaching α from both sides.

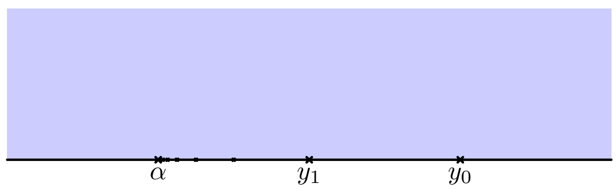


Figure 6.1: The vertices y_j.

When our boundary data is an infinite sum as described above, $\underline{w}$ has the form

$$\underline{w}(y) = \begin{cases} (m_0, n_0) & y > y_0 \\ (m_j, n_j) & y_{j+1} < y \le y_j \\ -(m_0, n_0) & y \le \alpha \end{cases} .$$

We also require that this function is non-singular and convex, and odd at infinity.

Then we can construct a scalar flat Kähler Joyce metric with the given boundary data. Since our boundary data is now only locally a step function it will be necessary to use a local construction to build the manifold on which we define our metrics, using the same local construction as Joyce [23], as seen in section (5.5).

Let M be the topological space whose orbit space is $\bar{\mathcal{H}}^2 \setminus \{(0, \alpha)\}$ with boundary data $\underline{w} : \mathbb{R} \to T(T^2)$. That is, take $(\bar{\mathcal{H}}^2 \setminus \{(0, \alpha)\}) \times T^2$ and at each point $y \in \mathbb{R}$ with $\underline{w}$ locally constant we contract the circle in T^2 orthogonal to $\underline{w}(y)$, and if $\underline{w}$ is not locally constant we contract T^2 at that point. At $(0, \infty)$, if the transformed boundary data

$$\underline{\tilde{w}}(y) = \operatorname{sign}(y)\underline{w}\left(-\frac{1}{y}\right)$$

is locally constant at 0 we contract the circle orthogonal to $\lim_{y\to 0} \underline{\tilde{w}}(y)$ and if not we contract the full T^2.

Then T^2 acts on this space by the standard action on the T^2 components, so that any point in $\partial\mathcal{H}^2$ with $\underline{w}$ locally constant is stabilised by $\underline{w}(y)^\perp$ and any point in $\partial\mathcal{H}^2$ with $\underline{w}$ not locally constant is a fixed point. This action is free on the orbit over every interior point, so the action is effective.

This gives us M as a topological space. However, if U is an open set in $\bar{\mathcal{H}}^2 \setminus \{(0, \alpha)\}$ containing only finitely many vertices then we can view $\underline{w}$ on U as a step function perturbed by a distribution supported outside of U, and considering charts

$$\begin{aligned} V &= (-\infty, \alpha) \cup \mathcal{H}^2 \\ U_i &= (\alpha + \frac{1}{i}, \infty) \cup \mathcal{H}^2 \end{aligned}$$

and a chart around $(0, \infty)$, it is clear the transition maps between U_i and V are smooth, giving $\pi^{-1}(\bar{\mathcal{H}}^2 \setminus \{(0, \alpha)\})$ the structure of a smooth manifold. Then if M is the resulting manifold,

$$g|_U = \frac{\rho \phi_1 \wedge \phi_2}{\sqrt{\rho^2 + (\eta - \alpha)^2}} \left(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\det(\phi_1, \cdot)^2 + \det(\phi_2, \cdot)^2}{(\phi_1 \wedge \phi_2)^2} \right)$$

is a scalar flat Kähler metric on M.

In order to prove this metric is complete, we will need to find a lower bound for $\phi_1 \wedge \phi_2$. Using this bound we will be able to show that the conformal factor grows sufficiently fast as we approach $(0, \alpha)$ that any curve approaching this point has infinite length under the Joyce metric, so the space is complete. This mirrors a remark of Joyce ([23], 3.3) that for a finite sum of basic solutions the metric near the base point of the conformal factor approaches that of $\mathbb{C}^2$ or $S^2 \times \mathcal{H}^2$ near ∞.

Theorem 6.1.1. *Let $\underline{w}$ be a distribution taking values among the primitive vectors, locally constant except at a collection of isolated points, $\{z_j\}_{j\in\mathbb{N}}$ with a single limit point $\alpha \in \partial\mathcal{H}^2 \setminus \{\infty\}$,*

$$\alpha < \ldots < z_1 < z_0$$

and denote

$$\underline{w}_j = \underline{w}(\eta) \qquad\qquad \forall \eta \in (z_{j+1},\, z_j)$$

(see diagram (6.1.1). Suppose that $\underline{w}$ *is convex and non-singular. Then the Joyce metric*

$$g = \frac{\rho \phi_1 \wedge \phi_2}{\sqrt{\rho^2 + (\eta - \alpha)^2}} \left(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\det(\phi_1,\, \cdot)^2 + \det(\phi_2,\, \cdot)^2}{(\phi_1 \wedge \phi_2)^2} \right)$$

is a complete scalar flat Kähler metric on M*, the manifold described above.*

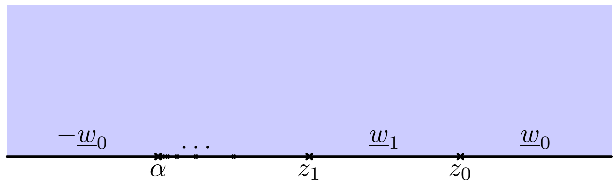


Figure 6.2: The boundary data function.

Proof. We prove this in several stages — first we find a lower bound on the pairings $\det(\underline{w}_i,\, \underline{w}_j)$, and use this to estimate $\phi_1 \wedge \phi_2$ as we approach the singular point. Using this estimate we can then find a lower bound on the length of a curve approaching the singularity, and thereby conclude that the space is complete.

Lemma 6.1.2. *Let* $\underline{w}$ *be as in (6.1.1). For each* $\underline{w}_j$*,*

$$\det(-\underline{w}_0,\, \underline{w}_j) \leq -1.$$

Proof. Since this determinant is a non-positive integer, we need only prove it is non-zero. Suppose for some $j \geq 1$

$$\det(-\underline{w}_0,\, \underline{w}_j) = 0$$

Since $\underline{w}$ takes values amongst the primitive elements of $\mathbb{Z}^2$, $\underline{w}_0 = \pm \underline{w}_j$. If $\underline{w}_0 = \underline{w}_j$,

$$\begin{aligned} -1 &= \det(\underline{w}_{j+1},\, \underline{w}_j) \\ &= \det(\underline{w}_{j+1},\, \underline{w}_0) \\ &= -\det(\underline{w}_0,\, \underline{w}_{j+1}) \end{aligned}$$

contradicting convexity. Similarly if $\underline{w}_0 = -\underline{w}_j$,

$$\begin{aligned} -1 &= \det(\underline{w}_j,\, \underline{w}_{j-1}) \\ &= -\det(\underline{w}_0,\, \underline{w}_{j-1}) \end{aligned}$$

again contradicting convexity. $\square$

This estimate allows us to bound the determinant by that of a solution with a non-singular vertex at $(0, \alpha)$ as follows:

Proposition 6.1.3. *Let (r, ξ) be polar coordinates about $(0, \alpha)$ and*

$$\begin{aligned}\phi_1 &= \int \frac{\rho(y-\eta)}{(\rho^2+(\eta-y)^2)^{\frac{3}{2}}} \underline{w}(y)\, dy \\ \phi_2 &= \int \frac{\rho^2}{(\rho^2+(\eta-y)^2)^{\frac{3}{2}}} \underline{w}(y)\, dy.\end{aligned}$$

If, for some $y_0 > \alpha$

$$\det(\underline{w}(y), \underline{w}(z)) \leq -1 \qquad \textit{for almost all } y < \alpha < z < y_0$$

then sufficiently close to $(0, \alpha)$, for some $C_1 > 0$

$$\phi_1 \wedge \phi_2 \geq C_1 \sin \xi.$$

Proof.

$$\begin{aligned}\phi_1 \wedge \phi_2 &= \int\int \frac{\rho^3(z-\eta)}{(\rho^2+(\eta-y)^2)^{\frac{3}{2}}(\rho^2+(\eta-z)^2)^{\frac{3}{2}}} \det(\underline{w}(y), \underline{w}(z))\, dydz \\ &= \int\int_{y\leq z} \frac{\rho^3(y-z)}{(\rho^2+(\eta-y)^2)^{\frac{3}{2}}(\rho^2+(\eta-z)^2)^{\frac{3}{2}}} \det(\underline{w}(y), \underline{w}(z))\, dydz \\ &\geq \int_\alpha^{y_0}\int_{-\infty}^{\alpha} \frac{\rho^3(z-y)}{(\rho^2+(\eta-y)^2)^{\frac{3}{2}}(\rho^2+(\eta-z)^2)^{\frac{3}{2}}}\, dydz,\end{aligned}$$

where we have symmetrized to obtain a positive integrand, then applied the bound on the determinant. Evaluating this integral explicitly we obtain

$$\phi_1 \wedge \phi_2 = \frac{\rho(y_0-\alpha)}{r\sqrt{\rho^2+(\eta-y_0)^2}} + \frac{\rho}{\sqrt{\rho^2+(\eta-\alpha)^2}} - \frac{\rho}{\sqrt{\rho^2+(\eta-y_0)^2}}.$$

For $r \leq \frac{y_0-\alpha}{2}$ the difference of the last two terms is positive, so that

$$\begin{aligned}\phi_1 \wedge \phi_2 &\geq \frac{\rho(y_0-\alpha)}{r\sqrt{\rho^2+(\eta-y_0)^2}} \\ &\geq \frac{\rho(y_0-\alpha)}{((y_0-\alpha)+r)r} \\ &\geq \frac{2\rho}{3r} = \frac{2}{3}\sin\xi.\end{aligned}$$

That is, we have compared the determinant near the singular point α with that near a non-singular vertex in a basic solution and found it must always be larger, and this gives us the bound we require. $\square$

If

$$U = \left\{ (\rho, \eta) \in \bar{\mathcal{H}}^2 | \sqrt{\rho^2 + (\eta - \alpha)^2} \leq \frac{y_0 - \alpha}{2} \right\},$$

then

$$\begin{aligned} g|_U &\geq \frac{\rho\phi_1 \wedge \phi_2}{r} \left(\frac{d\rho^2 + d\eta^2}{\rho^2} \right) \Bigg|_U \\ &\geq C_1 \frac{dr^2 + r^2 d\xi^2}{r^2} \Bigg|_U, \end{aligned}$$

where the inequalities are in the sense of (5.9). Hence any curve with image approaching $(0,\, \alpha)$ has infinite length. Therefore $(M,\, g)$ is complete. This completes the proof of (6.1.1). $\square$

We now relax the requirement that the vertices form a decreasing sequence, and instead ask that the set of vertices has only one accumulation point in $\partial\mathcal{H}^2$.

Corollary 6.1.4. *Suppose that $\underline{u}$ is a compactly supported distribution with derivative $\underline{w}$ which is a non-singular, convex function of the form*

$$\underline{w}(y) = \begin{cases} (m_0,\, n_0) & y > y_0 \\ (m_j,\, n_j) & y_j < y \leq y_{j-1}, \quad \forall j \geq 1 \\ (m_{-1},\, n_{-1}) & y < y_{-1} \\ (m_j,\, n_j) & y_{j+1} < y \leq y_j, \quad \forall j \leq -2 \end{cases}$$

and that $(m_{-1},\, n_{-1}) = -(m_0,\, n_0)$.

Then we can define the toric manifold M and the scalar flat Kähler manifold in the same way as in (6.1.1), and $(M,\, g)$ is complete.

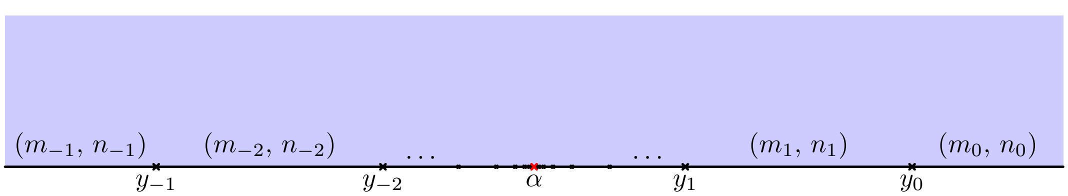


Figure 6.3: Boundary data with vertices approaching from both directions.

Proof. Note first that ∞ is not an accumulation point of $\{y_i\}_{i\in\mathbb{Z}}$. Then $\frac{dw}{dy}$ is compactly supported and the corresponding Joyce solution converges.

The assumption that the vertices are a decreasing sequence is only used in (6.1.2), so we just need to make sure this bound still holds. That is, that

$$\det(\underline{w}(y),\,\underline{w}(z)) \le -1 \qquad \text{for almost every } y < \alpha < z < y_0.$$

Then for any $\underline{w}_k$ with $k < 0$ we can apply the proof of (6.1.2) with $\underline{w}_k$ in place of $\underline{w}_0$ to show that $\det(\underline{w}_k,\,\underline{w}_j) \le -1$ for all $j \ge 0$ and the result follows. $\square$

This allows us to find complete scalar flat Kähler metrics on manifolds containing a string of spheres extending to infinity in both directions. In particular the metrics found in [26] are of this type, and can be found by setting the boundary data to be

$$\underline{w}(\eta) = \begin{cases} \cdots & \\ (k+1,\,k) & y_{k+1} < \eta < y_k \\ \cdots & \end{cases}$$

and allowing the positions of the vertices y_k to vary.

We can similarly extend this result to include a few more cases — firstly allowing the vertices to form an increasing sequence approaching α from below, and secondly allowing all but a finite number of vertices to form a monotonic sequence, increasing or decreasing, with the remaining vertices appearing on the opposite side of the singularity. However, the given results cover these cases after applying an isometry to hyperbolic space.

6.2 Self-dual Einstein metrics

Next we turn to the Einstein metrics and perform the analogous calculation. The proof of this follows similar lines to that of [6], and makes use of several of the bounds seen in section (5.6). However, we will extend the potential f_0 differently to $(-\infty,\,\alpha)$, and with this new potential (6.1.3) gives us a new lower bound on $\phi_1 \wedge \phi_2$ and allows us to remove the assumption that the e_j be bounded (using the notation of section (5.6)) and hence find many new metrics. We show at the end of the chapter how we may perturb this potential to find other extensions to f_0 for which the same proof holds.

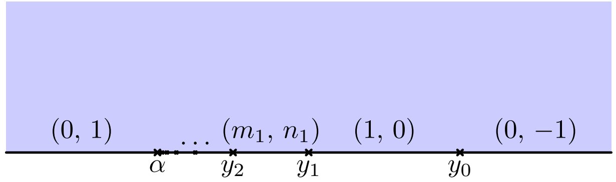


Figure 6.4: Boundary data for the SDE metric.

Suppose we have a Joyce solution

$$\phi = \sum f^{(y_i)} \otimes \underline{u}_i$$

and as in (6.1.1) the vertices $(y_i)_{i \in \mathbb{N}}$ form a decreasing sequence with $(y_i)_{i=1}^{\infty} \to \alpha$. Let the boundary data be

$$\underline{w}(y) = \begin{cases} (0,\,-1) & y_0 < y \\ (1,\,0) & y_1 < y < y_0 \\ (m_j,\,n_j) & y_j < y \leq y_{j-1},\ \forall j \geq 2 \\ (0,\,1) & y \leq \alpha \end{cases}$$

(see diagram (6.2)), and suppose the boundary data is convex and non-singular. Denote $(m_0,\,n_0) = (0,\,-1)$, $(m_1,\,n_1) = (1,\,0)$ and $(m_{-1},\,n_{-1}) = (0,\,1)$. If this is to satisfy (5.2.2), then we must have

$$\begin{aligned} y_j &= \frac{n_{j+1} - n_j}{m_{j+1} - m_j} \qquad\qquad j \geq 1 \\ y_0 &= 1. \end{aligned}$$

Set

$$e_j = m_{j+1} n_{j-1} - m_{j-1} n_{j+1},$$

suppose $e_j \geq 3\ \forall j \geq 1$ and define

$$f(\rho,\,\eta) = \int \frac{\rho^2 f_0(y)}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}}\, dy$$

as usual. Note that conversely, as in (5.3) and (5.6), we can instead start with the sequence $(e_j)_{j=1}^{\infty}$ and from this derive the vectors $(m_j,\,n_j)$ and vertices y_j to construct our boundary data.

This boundary data is convex and satisfies (5.2.2) so the Joyce conformal metric has an Einstein representative over

$$D^{+} = \{(\rho, \eta) \in \bar{\mathcal{H}^2} | f(\rho, \eta) > 0\}.$$

It also satisfies the hypotheses of (6.1.3), which allows us to bound $\phi_1 \wedge \phi_2$. We then need to determine the topology of D^+. Let

$$Z = \{(\rho, \eta) \in \mathcal{H}^2 | f(\rho, \eta) = 0\}.$$

Proposition 6.2.1. *With f and Z as above, Z consists of a smooth curve with exactly one component and its limit points are $(0, \alpha)$ and $(0, \infty)$.*

Proof. This result is given by [5], since we can think of this Joyce solution locally as a smeared solution. Then applying the proof of (5.2.5) gives the result. □

Now note that while our potential f_0 differs from that of (5.6.3) on $(-\infty, \alpha)$, the proof requires only that

$$f_0(y) \leq 0 \qquad \forall y < \alpha$$

so the same argument applies here, therefore there is some $D > 0$ such that

$$f(\rho, \eta) \leq D\sqrt{r} \qquad \forall (\rho, \eta) \in \mathcal{H}^2. \tag{6.1}$$

As in [6], we now consider a curve approaching the boundary of $\pi^{-1}(D^+)$, and show that this curve must have infinite length.

Proposition 6.2.2. *([6]) Let $\gamma : [0, 1) \to \pi^{-1}(D^+)$ be a curve with*

$$\lim_{t \to 1} \pi(\gamma(t)) = (\rho, \eta) \in Z.$$

or

$$\lim_{t \to 1} \pi(\gamma(t)) = (0, \infty) \in \partial\mathcal{H}^2.$$

Then

$$\text{length}\,\gamma = \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}\, dt = \infty.$$

Proof. This is proved by [6], as seen in (5.6.6) and (5.6.7). □

Finally we consider curves approaching the singularity. To show these have infinite length we use two bounds — first we have a lower bound on $\phi_1 \wedge \phi_2$ from (6.1.3) and second the upper bound for f from (6.1).

Proposition 6.2.3. *Let* $\gamma : [0,\, 1) \to \pi^{-1}(D^+)$ *be a curve with*

$$\lim_{t\to 1} \pi(\gamma(t)) = (0,\, \alpha).$$

Then

$$\text{length}\, \gamma = \int_0^1 \sqrt{g(\gamma'(t),\, \gamma'(t))}\, dt = \infty.$$

Proof. On some neighbourhood U_2 of $(0,\, \alpha)$ and for some $C_2 > 0$ we have from (6.1)

$$f(\rho,\, \eta) \leq C_2 r^{\frac{1}{2}},$$

where $(r,\, \xi)$ are radial coordinates about $(0,\, \alpha)$. By (6.1.3) these is a neighbourhood U_2 of $(0,\, \alpha)$ on which

$$\phi_1 \wedge \phi_2 \geq C_1 \sin \xi.$$

Then we have a lower bound for the metric on $U = U_1 \cap U_2$,

$$\begin{aligned} g|_U &\geq \left.\frac{\rho\phi_1 \wedge \phi_2}{f^2} \frac{dr^2 + r^2 d\xi^2}{r^2 \sin^2 \xi}\right|_U \\ &\geq \left.\frac{C_1}{C_2} \frac{dr^2 + r^2 d\xi^2}{r^2}\right|_U . \end{aligned}$$

Hence any curve γ approaching $(0,\, \alpha)$ has infinite length. $\square$

Having dealt with each of the three types of boundary points we have now proved the result:

Theorem 6.2.4. *If* ϕ *is an infinite sum of basic solutions satisfying (5.2.1) with convex, non-singular boundary data and*

$$\underline{w}(y) = \begin{cases} (0,\, -1) & y_0 < y \\ (1,\, 0) & y_1 < y < y_0 \\ (m_j,\, n_j) & y_j < y \leq y_{j-1},\ \forall j \geq 2 \\ (0,\, 1) & y \leq \alpha \end{cases}$$

where

$$\begin{aligned} y_j &= \frac{n_{j+1} - n_j}{m_{j+1} - m_j} \qquad\qquad j \geq 1 \\ y_0 &= 1. \end{aligned}$$

then

$$g = \frac{\phi_1 \wedge \phi_2}{F^2} \left(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\det(\phi_1, \cdot)^2 + \det(\phi_2, \cdot)^2}{(\phi_1 \wedge \phi_2)^2} \right)$$

is a complete Einstein metric on

$$\pi^{-1}(D_+) = \pi^{-1} \left(\{ (\rho, \eta) \in \mathcal{H}^2 | f(\rho, \eta) > 0 \} \right).$$

In [6], Calderbank-Singer note that under certain conditions, their complete Einstein metrics can be perturbed by a smeared solution to the left of the singularity to obtain new metrics. We can do likewise here, however, by examining the properties of the boundary data we have made use of, we are able to take perturbations right up to α.

Suppose $\underline{\tilde{w}}$ is a perturbation of $\underline{w}$ by a distribution supported on a compact set in $(-\infty, \alpha]$. If this perturbation is a locally constant function making the boundary data $\underline{\tilde{w}}$ convex, such that for some $y_0 > \alpha$

$$\det(\underline{\tilde{w}}(y), \underline{\tilde{w}}(z)) \leq -1 \qquad \text{for almost all } y < \alpha < z < y_0$$

then the bound from (6.1.3) still holds. In fact we only need that this quantity be bounded away from 0. Likewise, if $\underline{\tilde{w}}$ also satisfies (5.2.2), provided the corresponding potential $\tilde{f}_0$ given by (5.3) has

$$\tilde{f}_0(y) \leq 0 \qquad \forall y < \alpha$$

then (5.6.3) also holds. This guarantees that (5.6.6) and (6.2.3) hold, and since the perturbation is compactly supported it does not change the asymptotic values of f_0, so (5.6.7) is also true. However, for (6.2.1) to hold, we in fact need this inequality to be strict. Hence this result will extend to give a complete Einstein metric for such a perturbation too.

Corollary 6.2.5. *Let f_0 be the potential of a metric given by (6.2.4) with boundary data $\underline{w}$. Let $\tilde{f}_0$ be another potential with boundary data $\underline{\tilde{w}}$ such that:*

- *$\underline{w} - \underline{\tilde{w}}$ is a compactly supported step function on $(-\infty, \alpha]$.*

- *$\underline{\tilde{w}}$ is convex.*

- *For some $y_0 > \alpha$, $\lambda > 0$,*

$$\det(\underline{\tilde{w}}(y), \underline{\tilde{w}}(z)) \leq -\lambda \qquad \textit{for almost all } y < \alpha < z < y_0.$$

- *$\tilde{f}_0(y) < 0 \qquad \forall y < \alpha$.*

Then the self-dual Einstein metric with potential $\tilde{f}_0$ is complete.

Chapter 7

Local Joyce metrics

We have seen that we can obtain scalar flat Kähler metrics from solutions of the Joyce equations, and so far we have used linear combinations of basic solutions to construct a wide class of such metrics. However, it will be possible to obtain a second family of solutions by considering the Joyce equations directly. We then consider the problem of convergence of these solutions, and find criteria for when the positivity condition also holds, and hence construct a new family of Joyce metrics.

In order to do this, we first need to introduce a generalisation of Calderbank-Singer's results on smeared solutions (5.4.3, cf. [5]) which replaces the condition that the boundary data be convex with a more local condition. The resulting metrics no longer extend to all of $\mathcal{H}^2$, but will exist locally. This calculation plays an important role in stating the positivity condition for the local metrics.

We then apply these new solutions to some applications — first we find a Joyce form for the Ooguri-Vafa metric [19] and use this form to construct a large family of scalar flat Kähler perturbations of this metric. Secondly, we see how these new local solutions affect the metric on $\partial\mathcal{H}^2$. We apply this information to ask when we can find a Joyce metric on a torus fibration whose restriction to a subset of the degenerate orbits is prescribed.

7.1 Non-convex boundary data

So far we have assumed that all of our solutions are convex, so as to ensure that $\phi_1' \wedge \phi_2 > 0$ everywhere — however, in order to construct metrics locally it is possible to weaken this condition. We do this be revisiting the proof of (4.2.3) and calculating the asymptotic value of $\phi_1' \wedge \phi_2$ exactly.

Proposition 7.1.1. *Let ϕ be a linear combination of basic solutions with boundary data $\underline{w}$ locally constant at η_0. Then*

$$\phi_1' \wedge \phi_2(0,\, \eta_0) = \int \frac{\operatorname{sign}(z-\eta_0)}{(\eta_0 - z)^2} \det(\underline{w}(z),\, \underline{w}(\eta_0))\, dz.$$

Proof. There is some $\delta > 0$ such that $\underline{w}$ is constant on $U = (\eta_0 - \delta,\, \eta_0 + \delta)$. Then

$$\phi_1(\rho,\, \eta_0) = \int_U \frac{\rho(z-\eta_0)}{(\rho^2 + (\eta_0 - z)^2)^{\frac{3}{2}}}\, dz\, \underline{w}(\eta_0) + \int_{U^c} \frac{\rho(z-\eta_0)}{(\rho^2 + (\eta_0 - z)^2)^{\frac{3}{2}}} \underline{w}(z)\, dz.$$

The first term vanishes as it is symmetric about η_0, and since $\eta_0 - z$ is bounded away from 0 on U^c, we can apply the binomial expansion to the remaining term,

$$\phi_1 = \int_{U^c} \rho(z-\eta) \frac{1}{|\eta - z|^3} \left(1 + O\left(\frac{\rho^2}{(\eta - z)^2}\right)\right) \underline{w}(z)\, dz.$$

Since $\int \frac{1}{(\eta - z)^2}\, dz < \infty$, the error term passes through the integral,

$$\phi_1 = \int_{U^c} \rho \frac{\operatorname{sign}(z-\eta)}{(\eta - z)^2} \underline{w}(z)\, dz + O(\rho^2).$$

As we have seen, (4.2.2), $\phi_2 = \underline{w}(\eta_0) + O(\rho^2)$, so

$$\begin{aligned} \phi_1' \wedge \phi_2(\rho,\, \eta_0) &= \int_{U^c} \frac{\operatorname{sign}(z-\eta_0)}{(\eta_0 - z)^2} \det(\underline{w}(z),\, \underline{w}(\eta_0))\, dz + O(\rho^2) \\ &= \int \frac{\operatorname{sign}(z-\eta_0)}{(\eta_0 - z)^2} \det(\underline{w}(z),\, \underline{w}(\eta_0))\, dz + O(\rho^2). \end{aligned}$$

since the integrand is identically zero on U. ☐

Theorem 7.1.2. *If*

$$\begin{aligned} \phi_1 &= \int \frac{\rho(y-\eta)}{(\rho^2 + (\eta - y)^2)^{\frac{3}{2}}} \underline{w}(y) dy \\ \phi_2 &= \int \frac{\rho^2}{(\rho^2 + (\eta - z)^2)^{\frac{3}{2}}} \underline{w}(z) dz \end{aligned}$$

for $\underline{w}$ a distribution with compactly supported derivative, and $V \subseteq \partial\mathcal{H}^2 \setminus \{(0,\, \infty)\}$ an open set such that $\forall (0,\, y) \in V$, either

1. *$\underline{w}(y) = (m,\, n)$ is locally constant, and $(m,\, n)$ is primitive, or*

2.

$$\underline{w}(z) = \begin{cases} (m,\, n) & y < z \\ (m',\, n') & y > z \end{cases}$$

on some neighbourhood of $(0,\, y)$*, with* $\det \begin{pmatrix} m & m' \\ n & n' \end{pmatrix} = -1$

and if

$$\phi_1' \wedge \phi_2 = \int \frac{\operatorname{sign}(z-y)}{(y-z)^2} \det(\underline{w}(z),\, \underline{w}(y))\, dz > 0$$

for all points $(0,\, y) \in V$ *of the first kind then there is an open set* $U \subseteq \bar{\mathcal{H}}^2$ *with* $V = U \cap \partial\mathcal{H}^2$ *on which the Joyce metric generated by* ϕ *exists, and this metric extends to* $\pi^{-1}(U)$*.*

Proof. Theorem (4.2.5) required that $\phi_1' \wedge \phi_2$ extends to a positive function on the boundary. The previous proposition showed that at boundary points with $\underline{w}$ locally constant $\phi_1' \wedge \phi_2$ is precisely the given integral, so the asymptotic conditions (4.11) and (4.12) are satisfied. Then

$$U = V \cup \{(\rho,\, \eta) | \phi_1 \wedge \phi_2 > 0\}$$

is an open set, and (4.2.5) gives us the required Joyce metric on $\pi^{-1}(U)$. □

7.2 Local solutions from power series

In order to obtain formal solutions, we hypothesise a power series expansion for solutions of the Joyce equations and apply the Joyce equations to obtain conditions on the coefficients. We then find sufficient conditions for such a series to converge to obtain a genuine solution. Finally we examine the question of when we can perturb a sum of basic solutions with such a power series to obtain a solution satisfying the asymptotic conditions, and hence find a local Joyce metric.

Consider a solution (ϕ_1, ϕ_2) of the Joyce equations (4.1.2) on a simply connected neighbourhood $U \subseteq \mathcal{H}^2$. Since U is simply connected, there exists a $\mu : U \to \mathbb{R}^2$ such that

$$\begin{aligned} \phi_1 &= \frac{\partial \mu}{\partial \rho} \\ \phi_2 &= \frac{\partial \mu}{\partial \eta} \end{aligned}$$

and (as observed in [4]) solving these equations is equivalent to finding a function $\mu : U \to \mathbb{R}^2$ (defined up to an additive constant) such that

$$\rho \frac{\partial^2 \mu}{\partial \rho^2} + \rho \frac{\partial^2 \mu}{\partial \eta^2} - \frac{\partial \mu}{\partial \rho} = 0. \tag{7.1}$$

We refer to μ as the *Joyce potential* of this solution. Now suppose μ is an analytic function of ρ satisfying this condition

$$\mu(\rho, \eta) = \sum_{i=0}^{\infty} f_i(\eta) \rho^i.$$

Then (7.1) becomes

$$\sum_{i=0}^{\infty} \left(i(i+2) f_{i+2}(\eta) + f_i''(\eta) \right) \rho^{i+1} - f_1(\eta) = 0.$$

Solving this term by term gives

$$\begin{aligned} f_1(\eta) &= 0 \\ f_0''(\eta) &= 0 \\ i(i+2) f_{i+2}(\eta) + f_i''(\eta) &= 0, \ \forall i \geq 1. \end{aligned}$$

and hence

$$\begin{aligned} f_{2i+1}(\eta) &= 0 & \forall i \geq 0 \\ f_0(\eta) &= a\eta + b & \\ f_{2i+2}(\eta) &= -\tfrac{f_{2i}''(\eta)}{2i(2i+2)} & \forall i \geq 1 \end{aligned}$$

Conversely, given a smooth function $f : \mathbb{R} \to \mathbb{R}^2$, $a,\, b \in \mathbb{R}^2$ we can write down a formal power series solving (7.1),

$$\mu = a\eta + b + \sum_{i=0}^{\infty} (-1)^i \frac{f^{(2i)}(\eta)}{2^{2i} i!(i+1)!} \rho^{2i+2}. \tag{7.2}$$

Whenever this power series is absolutely convergent, μ is a genuine function and solves (7.1).

Now suppose we are given constants $a,\, b \in \mathbb{R}^2$ and a smooth function $f : \mathbb{R} \to \mathbb{R}^2$ such that for some $\lambda > 0$ and some open set $V \subseteq \mathbb{R}$,

$$|f^{(2i)}(\eta)| \leq 2^{2i} i!(i+1)!\lambda^{i+1} \qquad\qquad \forall i \geq 0,\, \eta \in V. \tag{7.3}$$

Then the corresponding formal solution has

$$\begin{aligned} |\mu(\eta)| \quad &\leq \quad |a + b\eta| + \sum_{i=0}^{\infty} \frac{|f^{(2i)}(\eta)|}{2^{2i} i!(i+1)!} \rho^{2i+2} \\ &\leq \quad |a + b\eta| + \sum_{i=0}^{\infty} \left(\lambda \rho^2\right)^{i+1}. \end{aligned}$$

In particular, on the region

$$U = \{(\rho,\, \eta) | \rho < \frac{1}{2}\sqrt{\lambda}\} \tag{7.4}$$

μ converges uniformly and we have a Joyce solution $\phi = \left(\frac{\partial \mu}{\partial \rho},\, \frac{\partial \mu}{\partial \eta}\right)$.

Now let $(\tilde{\phi}_1,\, \tilde{\phi}_2)$ be a non-singular sum of basic solutions with boundary data $\underline{w}$. The corresponding solution of (7.1) is

$$\tilde{\mu} = \int \sqrt{\rho^2 + (\eta - y)^2}\, \underline{w}(y)\, dy.$$

Then

$$\mu = \tilde{\mu} + \sum_{i=0}^{\infty} \frac{(-1)^i f^{(2i)}(\eta)}{2^{2i} i!(i+1)!} \rho^{2i+2} \tag{7.5}$$

is a solution of (7.1) and we examine the question of when the Joyce solution $\left(\frac{\partial \mu}{\partial \rho},\, \frac{\partial \mu}{\partial \eta}\right)$ defines a Joyce metric on some suitable set. Here we assume without loss of generality that $a = b = 0$, since they correspond to the solution

$$\phi_1 = 0,\, \phi_2 = a$$

and a constant of integration respectively, and hence can be absorbed into $\tilde{\mu}$.

To see that this gives a Joyce metric, first we must consider the asymptotic behaviour of this solution to show that (4.11) and (4.12) are satisfied:

Lemma 7.2.1. *For the above solution (7.5), if y is an edge point of $\underline{w}$ then*

$$\phi_1(\rho,\,\eta) = O(\rho)$$
$$\phi_2(\rho,\,\eta) = \underline{w}(y) + O(\rho^2)$$

and if y is a vertex of $\underline{w}$,

$$\phi_1(\rho,\,z) = \tfrac{\rho}{2r}((m,\,n) - (m',\,n')) + O(r)$$
$$\phi_2(\rho,\,z) = \tfrac{1}{2}((m,\,n) + (m',\,n')) + \tfrac{\eta - y}{r}\tfrac{1}{2}((m,\,n) - (m',\,n')) + O(r^2).$$

Proof.

$$\begin{aligned}
\phi_1 &= \tilde{\phi}_1 + \frac{\partial \mu}{\partial \rho} \\
&= \tilde{\phi}_1 + \rho \sum_{i=0}^{\infty} \frac{(-1)^i f^{(2i)}(\eta)}{2^{2i-1}(i!)^2} \rho^{2i} \\
\phi_2 &= \tilde{\phi}_2 + \frac{\partial \mu}{\partial \eta} \\
&= \tilde{\phi}_2 + \rho^2 \sum_{i=0}^{\infty} \frac{(-1)^i f^{(2i+1)}(\eta)}{2^{2i} i!(i+1)!} \rho^{2i}.
\end{aligned}$$

and the two sums converge on U. Here we may differentiate term by term since the sum is uniformly convergent. Then the perturbation vanishes to high enough order that it does not contribute to these asymptotic conditions. □

It remains to show when the determinant $\phi_1' \wedge \phi_2$ is positive as we approach the boundary.

Proposition 7.2.2. *With the above solution (7.5), if $V \subseteq \partial\mathcal{H}^2 \setminus \{(0,\,\infty)\}$ is an open set such that there exists a $\delta > 0$ with $\forall \eta \in V$*

$$\tilde{\phi}_1' \wedge \tilde{\phi}_2 + 2\det\left(f(\eta),\,\underline{w}(\eta)\right) > \delta$$

then there exists an open set $\tilde{U} \subseteq \bar{\mathcal{H}}^2$ with $\tilde{U} \cap \partial\mathcal{H}^2 = V$ such that

$$\phi_1' \wedge \phi_2 > 0 \qquad\qquad \forall(\rho,\,\eta) \in \tilde{U} \cap \mathcal{H}^2$$

Proof. As in the previous lemma

$$\begin{aligned}\phi_1 &= \tilde{\phi}_1 + 2\rho f(\eta) + \rho^3 \sum_{k=1}^{\infty} \frac{(-1)^k f^{(2k+1)}(\eta)}{2^{2k-1}(k!)^2} \rho^{2k-2} \\ \phi_2 &= \tilde{\phi}_2 + \rho^2 \sum_{k=0}^{\infty} \frac{(-1)^k f^{(2k+1)}(\eta)}{2^{2k} k!(k+1)!} \rho^{2k+2}\end{aligned}$$

Hence

$$\phi_1 \wedge \phi_2 = \tilde{\phi}_1 \wedge \tilde{\phi}_2 + 2\rho \det(f(\eta), \underline{w}(\eta)) + O(\rho^2) \tag{7.6}$$

□

Note that we can use (7.1.1) to find the first term. In particular, if the boundary data is convex the first term is everywhere positive, in which case, given any smooth bounded h, since $\tilde{\mu}_\eta(\eta) = \underline{w}(\eta)$ is bounded, there is some $\gamma > 0$ such that putting $f = \gamma h$ satisfies the positivity condition on U.

Putting these results together:

Theorem 7.2.3. *Let*

$$\mu = \tilde{\mu} + \sum_{i=0}^{\infty} \frac{(-1)^i f^{(2i)}(\eta)}{2^{2i} i!(i+1)!} \rho^{2i+2}$$

with $V \subseteq \mathbb{R}$ such that for some $\lambda > 0$

$$|f^{(2i)}(\eta)| \leq 2^{2i} i!(i+1)! \lambda^{i+1} \qquad \forall i \geq 0,\, \eta \in V$$

and for some $\delta > 0$ such that

$$\tilde{\phi}_1' \wedge \tilde{\phi}_2 + 2 \det\left(f(\eta), \underline{w}(\eta)\right) > \delta \qquad \forall \eta \in V.$$

Then on some open set $U \subseteq \bar{\mathcal{H}}^2$ with $V = U \cap \partial\mathcal{H}^2$, $\left(\frac{\partial \mu}{\partial \rho}, \frac{\partial \mu}{\partial \eta}\right)$ is a Joyce solution and yields a Joyce metric on U.

7.3 The Ooguri-Vafa metric

We saw in section (2.4) that the Ooguri-Vafa metric is a periodic S^1-invariant hyperkähler metric obtained from the Gibbons-Hawking ansatz and that this space is useful as a model for degenerate fibres of elliptic fibrations. However, the rigidity of hyperkähler metrics makes this approach quite inflexible. Rediscovering the Ooguri-

Vafa metric as a Joyce solution will make it possible to find scalar flat Kähler, but not hyperkähler, perturbations which cannot be found from the Gibbons-Hawking ansatz.

We noted in section (2.4) in the space of Ooguri-Vafa the degenerate orbits form a chain of spheres, each intersecting its neighbour at a single point. The adjunction formula (see [18], for example) tells us that each of these spheres has self-intersection 2, and we are able to use this information to deduce boundary data, which will allow us to find the Ooguri-Vafa metric. We then verify this by explicit calculation. This construction is similar to Joyce's non-simply connected metrics ([23], cf. section 5.5), which use an infinite sum of basic solutions to build a periodic metric. However, the local solutions of the previous sections will play a crucial role here, ensuring the convergence of the periodic solution.

Proposition 7.3.1. *A Joyce metric invariant under integer translations in η whose boundary consists of a chain of spheres, each having self-intersection* 2*, must have boundary data*

$$\underline{w}(\eta) = \begin{cases} \dots & \\ (k+1,\, k) & -1-k < \eta \leq -k \\ \dots & \end{cases}$$

up to isometries of hyperbolic space and change of basis in $\mathbb{R}^2$.

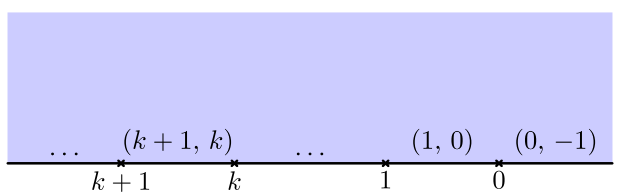


Figure 7.1: The boundary data.

Proof. We can assume the set of vertices is $\mathbb{Z}$, so that $\underline{w}$ has the form

$$\underline{w}(\eta) = \begin{cases} \dots & \\ (a_k,\, b_k) & -1-k < \eta \leq -k \\ \dots & \end{cases}$$

We then fix a basis so that $(a_0,\, b_0) = (1,\, 0)$, $(a_{-1},\, b_{-1}) = (0,\, -1)$, which is possible since $\det((a_0,\, b_0),\, (a_{-1},\, b_{-1})) = -1$.

Now, since each vertex is a smooth point, we have

$$\det((a_{k+1},\, b_{k+1}),\, (a_k,\, b_k)) = -1 \qquad\qquad \forall k \in \mathbb{Z}$$

and the self-intersection of the kth sphere (see [5], p415) is given by

$$-\det((a_{k+1},\, b_{k+1}),\, (a_{k-1},\, b_{k-1})) = 2 \qquad\qquad \forall k \in \mathbb{Z}.$$

Suppose $(a_{k-1},\, b_{k-1}) = (k,\, k-1)$, $(a_k,\, b_k) = (k+1,\, k)$. Then

$$\begin{aligned} -1 &= a_{k+1}b_k - a_k b_{k+1} = k a_{k+1} - (k+1) b_{k+1} \\ -2 &= a_{k+1}b_{k-1} - a_{k-1}b_{k+1} = (k-1)a_{k+1} - k b_{k+1} \end{aligned}$$

and solving these gives $(a_{k+1},\, b_{k+1}) = (k+2,\, k+1)$, and by induction this then holds for all $\forall k \geq 0$. A similar argument gives the same result for $k < 0$. □

Then the corresponding Joyce potential has the form

$$\begin{aligned} \mu &= \sum_{k\in\mathbb{Z}} \Big(\sqrt{\rho^2 + (\eta - k)^2} \otimes \frac{1}{2}(1,\, 1) + a_k + b_k \eta + && (7.7) \\ &\qquad + \rho^2 c_k(\eta) + O(\rho^4) \Big) + \eta \otimes (1,\, 0). && (7.8) \end{aligned}$$

However, we must ensure this sum converges. By the binomial expansion,

$$\begin{aligned} \sqrt{\rho^2 + (\eta - k)^2} &= |k| \sqrt{1 + \left(\frac{\rho^2}{k^2} + \frac{\eta^2}{k^2} - 2\frac{\eta}{k} \right)} \\ &= |k| + \frac{\rho^2}{2|k|} - 2\eta \,\mathrm{sign}\, k + O(k^{-2}) \end{aligned}$$

for sufficiently large k. In particular, if we put

$$\begin{aligned} a_k &= -\frac{|k|}{2}(1,\, 1) \\ b_k &= \mathrm{sign}(k)(1,\, 1) \\ c_k(\eta) &= -\frac{1}{4|k|}(1,\, 1) \end{aligned}$$

the sum will converge uniformly on every compact set in $\mathbb{R}$.

Theorem 7.3.2. *Take potential (7.7) with*

$$\begin{aligned} a_k &= -\frac{|k|}{2}(1,\, 1) \\ b_k &= \operatorname{sign}(k)(1,\, 1) \\ c_k(\eta) &= -\frac{1}{4|k|}(1,\, 1) \end{aligned}$$

where

$$\lambda_k = \begin{cases} 0 & k = 0 \\ \frac{1}{|k|} & k \neq 0 \end{cases}$$

The Ooguri-Vafa metric is one of the scalar flat Kähler metrics with Joyce potential μ.

Proof. Calculating the Joyce solution,

$$\begin{aligned} \phi_1 &= -\frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\frac{\rho}{\sqrt{\rho^2+(\eta-k)^2}} - \lambda_k\rho\right)\otimes(1,\, 1) \\ \phi_2 &= -\frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\frac{(\eta-k)}{\sqrt{\rho^2+(\eta-k)^2}} + \operatorname{sign}(k)\right)\otimes(1,\, 1) + (1,\, 0). \end{aligned}$$

Then we can calculate

$$\begin{aligned} \phi_1\wedge\phi_2 &= \frac{\rho}{2}\sum_{k\in\mathbb{Z}}\left(\frac{1}{\sqrt{\rho^2+(\eta-k)^2}} - \lambda_k\right) \\ &= \frac{\rho V}{2}. \end{aligned}$$

Identifying

$$(r,\, u_2,\, \xi,\, t) = (\rho,\, \eta,\, \theta_2-\theta_1,\, 2\pi\theta_2)$$

we find that

$$\begin{aligned} \det(\phi_1,\, d\underline{\theta}) &= -\frac{\rho}{2}\sum_{k\in\mathbb{Z}}\left(\frac{1}{\sqrt{\rho^2+(\eta-k)^2}} - \lambda_k\right)(d\theta_2 - d\theta_1) = -\frac{\rho V}{2}d\xi \\ \det(\phi_2,\, d\underline{\theta}) &= -\frac{1}{2}\sum_{k\in\mathbb{Z}}\left(\frac{(\eta-k)}{\sqrt{\rho^2+(\eta-k)^2}} + \operatorname{sign}(k)\right)(d\theta_2 - d\theta_1) - \frac{1}{2}d\theta_2 = -\frac{1}{2}\theta_0, \end{aligned}$$

where θ_0 is as in (2.1), seen in section (2.4). Substituting these expressions into the

Joyce scalar flat Kähler metric and comparing with (2.2) gives

$$\begin{aligned} g_J &= \rho\phi_1 \wedge \phi_2 \left(\frac{d\rho^2 + d\eta^2}{\rho^2} + \frac{\det(\phi_1, d\underline{\theta})^2 + \det(\phi_2, d\underline{\theta})^2}{(\phi_1 \wedge \phi_2)^2} \right) \\ &= \frac{V}{2}(d\rho^2 + d\eta^2 + \rho^2 d\xi^2) + \frac{1}{2V}\theta_0^2 \\ &= \frac{1}{2} g_{OV}. \end{aligned}$$

□

Theorem 7.3.3. *Let f be a smooth periodic function with*

$$|f^{(2i)}(\eta)| \leq 2^{2i} i!(i+1)!\lambda^i \qquad \forall i \geq 0,\, \eta \in [0,\, 1],$$

and μ the potential of the previous theorem. Then for some $\gamma > 0$ the Joyce solution

$$\tilde{\mu} = \mu + \gamma \sum_{i=0}^{\infty} (-1)^i \frac{f^{(2i)}(\eta)}{2^{2i} i!(i+1)!}$$

gives a Joyce metric on a neighbourhood of the boundary, and the Joyce metric with Joyce solution $\left(\frac{\partial \mu}{\partial \rho}, \frac{\partial \mu}{\partial \eta}\right)$ and conformal factor $\Omega^2 = \rho\phi_1 \wedge \phi_2$ is a scalar flat Kähler perturbation of the Ooguri-Vafa metric on this neighbourhood.

Proof. The first condition guarantees convergence on a region $\{(\rho,\, \eta)\,|\, \rho < \sqrt{\lambda}\}$ by (7.4). Since f is smooth it is bounded on compact intervals, and being periodic is then globally bounded. Then as we remarked at the end of section (7.2), for sufficiently small γ we can find a neighbourhood of $[0,\, 1] \subseteq \partial\mathcal{H}^2$ on which $\tilde{\mu}_\rho \wedge \tilde{\mu}_\eta > 0$, and we can extend this set periodically to obtain a neighbourhood of $\partial\mathcal{H}^2 \setminus \{(0,\, \infty)\}$ on which the Joyce metric is defined. □

In this way we can obtain a large family of perturbations of the Ooguri-Vafa metric. In fact, using a result in the following section, (7.4.2), it is shown that for this inequality to hold it is sufficient that f be analytic with radius of convergence uniformly bounded away from zero. Then, for example, any trigonometric polynomial will satisfy these conditions.

7.4 Prescribing the metric on the central fibre

One application of these new Joyce metrics will be to a problem suggested by Joel Fine, concerning constructing Kähler metrics on fibrations of complex surfaces by compact complex surfaces [12], as follows:

We view $\mathbb{C}^2$ as a fibration of hyperboloids over $\mathbb{C}$,

$$\pi : \mathbb{C}^2 \to \mathbb{C} \qquad\qquad \pi(w,\, z) = wz,$$

and examine a region close to the origin in this space. The central fibre,

$$C = \{(w,\, z) | wz = 0\},$$

is then degenerate and provides a good model of singular points on the typical degenerate fibres which appear in more general fibrations by Riemann surfaces. Then

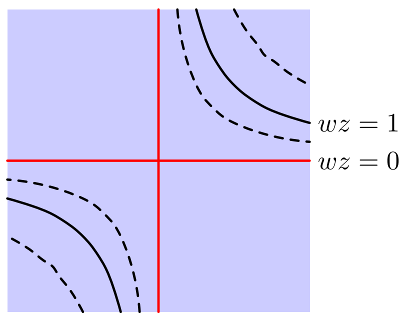


Figure 7.2: The real part of the fibration of $\mathbb{C}^2$ by hyperboloids.

scalar flat Kähler metrics on neighbourhoods of the origin will provide good models for Kähler metrics on fibrations near singular points. For this reason, we attempt to construct toric scalar flat Kähler metrics on neighbourhoods of the origin in $\mathbb{C}^2$ whose restriction to the central fibre is prescribed on an annulus. In particular we require that its restriction should be the cusp metric.

More precisely, we attempt to construct Joyce metrics on a neighbourhood of an interval $(-\delta_2,\, \delta_2) \subseteq [-1,\, 1]$ with boundary data

$$\underline{w}(\eta) = \begin{cases} (1,\, 0) & 0 < \eta \leq 1 \\ (0,\, 1) & -1 \leq \eta \leq 0 \\ (0,\, 0) & \text{otherwise.} \end{cases} \tag{7.9}$$

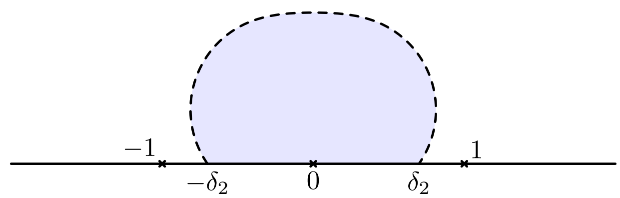
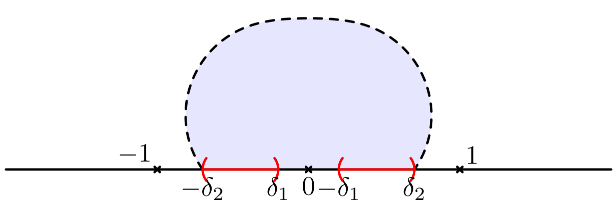
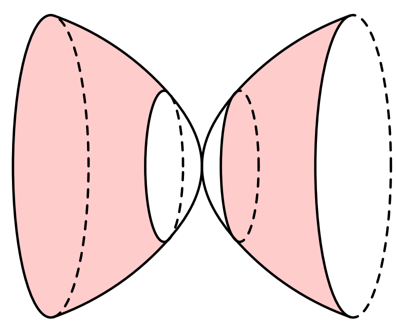


Figure 7.3:

This condition describes the torus fibration on which we construct Joyce metrics — close to the origin, the space consists of a disc fibred by hyperboloids given by the contours $\eta = c$, $\theta_1 + \theta_2 = \alpha$ where $(c,\, \alpha)$ are polar coordinates on the disc. The degenerate fibre, $\{\eta = 0\}$, consists of two discs meeting at a single point, with the discs corresponding to the regions $\{\eta \geq 0\}$ and $\{\eta \leq 0\}$.



(a) The region on which the metric exists.



(b) The topology of the central fibre.

Note that while we cannot use topological conditions to fix the boundary data outside $[-1,\, 1]$, the effect of changing this data only changes the terms of higher order in ρ, so that varying this data is equivalent to perturbing by a different local metric. Hence we can make this choice without loss of generality.

We then require that for some $0 < \delta_1 < \delta_2$ the Joyce metric restricts to

$$g_J|_{\pi^{-1}(\{0\}\times(\delta_1,\, \delta_2))} = \frac{1}{\eta^2} d\eta^2 + \eta^2 d\theta_2^2$$

and

$$g_J|_{\pi^{-1}(\{0\}\times(-\delta_2,\,-\delta_1))} = \frac{1}{\eta^2}d\eta^2 + \eta^2 d\theta_1^2,$$

the cusp metric on each of the two components of the central fibre, restricted to an annulus around the origin.

In order to do this, we first calculate the restriction of the Joyce metric to the central fibre, which will give us a restriction on the possible functions f from which we can build a potential. By reconsidering the constraint (7.3) we then show that no such local solution can converge on this region. Finally, we show that while no exact solution is possible, we can construct a sequence of metrics whose restrictions converge to the cusp metrics on the given region.

To calculate the metric on the central fibre, let I be an interval on which $\underline{w}(\eta) = \underline{w}$ is constant, for a Joyce solution $(\phi_1,\, \phi_2)$ with potential

$$\mu = \int \sqrt{\rho^2 + (\eta - y)^2}\,\underline{w}(y)\, dy + \sum_{i=0}^{\infty} \frac{f^{(2i)}(\eta)}{2^{2i} i!(i+1)!}\rho^{2i+2} \tag{7.10}$$

and let

$$\tilde{\mu} = \int \sqrt{\rho^2 + (\eta - y)^2}\,\underline{w}(y)\, dy.$$

If we denote

$$\epsilon'(\eta) = \phi_1' \wedge \phi_2(0,\, \eta),$$

then the Joyce metric restricts to

$$g_J|_{\pi^{-1}(\{0\}\times I)} = \epsilon' d\eta^2 + \frac{1}{\epsilon'}\det(\underline{w},\, d\underline{\theta})^2.$$

In particular, the restriction of the metric is completely determined by $\underline{w}$ and ϵ'.

Then in order to obtain our desired restrictions we should find a solution with

$$\epsilon'(\eta) = \frac{1}{\eta^2} \qquad\qquad \forall \eta \in (-\delta_2,\, -\delta_1) \cup (\delta_1,\, \delta_2).$$

Then from (7.6) we must have

$$2\det(f(\eta),\, \underline{w}(\eta)) = \frac{1}{\eta^2} - \tilde{\phi}_1' \wedge \tilde{\phi}_2.$$

We can calculate this second term explicitly using (7.1.1),

Lemma 7.4.1. *If $(\tilde{\phi}_1, \tilde{\phi}_2)$ is the sum of basic solutions with boundary data (7.9), then*

$$\tilde{\phi}_1 \wedge \tilde{\phi}_2 = \begin{cases} \frac{1}{\eta(1+\eta)} & 0 < \eta < 1 \\ \frac{-1}{\eta(1-\eta)} & -1 < \eta < 0 \end{cases}$$

and if μ is as above, (7.10) and has

$$\phi_1' \wedge \phi_2(0, \eta) = \begin{cases} \frac{1}{\eta^2} & \delta_1 < \eta < \delta_2 \\ \frac{1}{\eta^2} & -\delta_2 < \eta < -\delta_1 \end{cases}$$

then

$$\begin{array}{ll} 2f_2(\eta) = \frac{1}{\eta^2} - \frac{1}{\eta(1+\eta)} & \delta_1 < \eta < \delta_2 \\ -2f_2(\eta) = \frac{1}{\eta^2} + \frac{1}{\eta(1-\eta)} & -\delta_2 < \eta < -\delta_1 \end{array}.$$

Then the components of f must be smooth functions extending from these values to all of $(-\delta_2, \delta_2)$, and in particular over $\eta = 0$. Any such f will give us a formal solution, but we must also check that such a solution converges.

Proposition 7.4.2. *Let f be a smooth function for which (7.3) holds on I, then $\exists \tilde{\lambda} > 0$ such that*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(\eta_0)(\eta - \eta_0)^k}{k!} \quad \textit{converges } \forall |\eta - \eta_0| < \lambda,\ \eta_0 \in I$$

That is, the radius of convergence of f is uniformly bounded below on I.

Proof.

$$\begin{aligned} |f^{(2k)}(\eta_0)| &\leq 2^{2k} k!(k+1)!\lambda^{2k} \\ &\leq 2^k k! \left(\prod_{j=1}^{k} (2j+1) \right) \lambda^{2k} \\ &= (2k)!((2k+1)(2k+3)\lambda^{2k}) \\ &\leq (2k)!\tilde{\lambda}^{2k} \end{aligned}$$

where $\lambda < \tilde{\lambda}$ is large enough that

$$(2k+1)(2k+3) \leq \left(\frac{\tilde{\lambda}}{\lambda} \right)^{2k} \qquad \forall k \in \mathbb{N}.$$

In order to bound the odd terms we will need an extra technical result, which uses bounds on the even derivatives of a function to control the growth of the odd terms:

Lemma 7.4.3. *Take $f : [a, b] \to \mathbb{R}$ a smooth function such that*

$$|f(x)| \leq A, \qquad |f''(x)| \leq C \qquad \forall x \in (a, b)$$

and suppose that $\frac{(b-a)}{2} > 2\sqrt{\frac{A}{C}}$. Then

$$|f'(x)| \leq 2\sqrt{AC} \qquad \forall x \in (a, b).$$

Proof. Suppose for some $x_0 \in (a, b)$ that $f'(x_0) > 2\sqrt{AC}$. Since the interval is large enough, either $x_0 + 2\sqrt{\frac{A}{C}} \leq b$ or $x_0 - 2\sqrt{\frac{A}{C}} \geq a$. We assume without loss of generality the former holds. Then

$$\begin{aligned}
\left| f\left(x_0 + 2\sqrt{\frac{A}{C}}\right) - f(x_0) \right| &= \left| \int_{x_0}^{x_0 + 2\sqrt{\frac{A}{C}}} f'(x)\, dx \right| \\
&\geq \left| \int_{x_0}^{x_0 + 2\sqrt{\frac{A}{C}}} f'(x_0) + (x - x_0) f''(x)\, dx \right| \\
&> \int_0^{2\sqrt{\frac{A}{C}}} 2\sqrt{AC} - xC\, dx = 2A.
\end{aligned}$$

This violates the bound of f, so no such x_0 can exist. ☐

Take $i \in \mathbb{N}$. We have

$$|f^{(2k)}(\eta)| \leq (2k)!\tilde{\lambda}^{2k} \text{ and } |f^{2k+2}(\eta)| \leq (2k+2)!\tilde{\lambda}^{2k+2} \qquad \forall \eta \in (-\delta_2, \delta_2)$$

so for k sufficiently large that

$$\delta_2 > \frac{2}{\sqrt{(2k+1)(2k+2)}}$$

the lemma gives us

$$\begin{aligned}
|f^{(2k+1)}(\eta_0)| &\leq 2\sqrt{(2k+1)(2k+2)}(2k)!\tilde{\lambda}^{2k+1} \\
&\leq 4(2k+1)!\tilde{\lambda}^{2k+1}.
\end{aligned}$$

In particular,

$$\sum f^{(2k+1)}(\eta_0)\frac{(\eta-\eta_0)^{2k+1}}{(2k+1)!} < \infty \qquad \forall \eta_0 \in (-\delta_2,\, \delta_2),\ |\eta-\eta_0| < \frac{1}{\tilde{\lambda}},$$

since the sums of odd and even terms converge absolutely, so does the full sum. $\square$

Any f_1 and f_2 satisfying the required conditions must be non-analytic at at least one point in $(-\delta_1,\, \delta_1)$. However, this bound implies that f must be analytic, hence no solution can exist:

Theorem 7.4.4. *There are no $0 < \delta_1 < \delta_2 < 1$ with a Joyce potential*

$$\mu = \int \sqrt{\rho^2 + (\eta - y)^2}\underline{w}(y)\, dy + \sum_{i=0}^{\infty} \frac{f^{(2i)}(\eta)}{2^{2i} i!(i+1)!}\rho^{2i+2}$$

on an open set U with $U \cap \bar{\mathcal{H}}^2 = (-\delta_2,\, \delta_2)$, where

$$\underline{w}(\eta) = \begin{cases} (1,\, 0) & 0 < \eta \leq 1 \\ (0,\, 1) & -1 \leq \eta \leq 0 \\ (0,\, 0) & \textit{otherwise}, \end{cases}$$

whose Joyce metric restricts to the cusp metric on an annulus intersected with the central fibre,

$$\begin{aligned} g_J|_{\pi^{-1}(\{0\}\times(\delta_1,\delta_2))} &= \frac{1}{\eta^2}d\eta^2 + \eta^2 d\theta_2^2 \\ g_J|_{\pi^{-1}(\{0\}\times(-\delta_2,-\delta_1))} &= \frac{1}{\eta^2}d\eta^2 + \eta^2 d\theta_1^2. \end{aligned}$$

While we will not be able to find a metric restricting to exactly the cusp metric on the two annuli, it will be possible to construct sequences approximating it.

Proposition 7.4.5. *There are sequences of polynomials, A_k, B_k on neighbourhoods U_k of $(-\delta_2,\, \delta_2)$ such that the Joyce metric*

$$\mu = \int \sqrt{\rho^2 + (\eta - y)^2}\underline{w}(y)\, dy + \sum_{i=0}^{\infty} \frac{f^{(2i)}(\eta)}{2^{2i} i!(i+1)!}\rho^{2i+2},$$

where

$$\underline{w}(\eta) = \begin{cases} (1,\, 0) & 0 < \eta \leq 1 \\ (0,\, 1) & -1 \leq \eta \leq 0 \\ (0,\, 0) & \textit{otherwise} \end{cases}$$

and $f = (A_k,\, B_k)$, *has restriction to*

$$\pi^{-1}(\{0\} \times (-\delta_2,\, -\delta_1)) \text{ and } \pi^{-1}(\{0\} \times (\delta_1,\, \delta_2))$$

which converges C^0 *uniformly to the cusp metric.*

Proof. Choose $0 < \alpha < \delta$ sufficiently small that

$$\begin{aligned} \frac{d}{d\eta}\left(\frac{1}{\eta(1+\eta)}\right) &< 0 \quad \forall \eta \in (0,\, \alpha) \\ \frac{d}{d\eta}\left(\frac{1}{\eta(1-\eta)}\right) &> 0 \quad \forall \eta \in (-\alpha,\, 0) \end{aligned}$$

and define

$$\begin{aligned} A(\eta) &= \begin{cases} \frac{1}{\eta^2} - \left(\frac{1}{\eta(1+\eta)}\right) & 1 > \eta \geq \alpha \\ \frac{1}{\alpha^2} - \left(\frac{1}{\alpha(1+\alpha)}\right) & \eta < \alpha \end{cases} \\ B(\eta) &= \begin{cases} -\frac{1}{\eta^2} + \left(\frac{1}{\eta(1-\eta)}\right) & -1 < \eta \leq -\alpha \\ -\frac{1}{\alpha^2} - \left(\frac{1}{\alpha(1-\alpha)}\right) & \eta > -\alpha. \end{cases} \end{aligned}$$

Then from the bound on the derivatives

$$\begin{aligned} \left(\tfrac{1}{\eta(1+\eta)}\right) + A(\eta) \geq \tfrac{1}{\alpha^2} > 0 \qquad & \forall \eta < \alpha \\ \left(\tfrac{1}{\eta(1-\eta)}\right) - B(\eta) \geq \tfrac{1}{\alpha^2} > 0 \qquad & \forall \eta > -\alpha. \end{aligned} \tag{7.11}$$

Now let (A_k) and (B_k) be sequences of polynomials uniformly approximating A and B on $[0,\, \delta_1]$ and $[-\delta_1,\, 0]$ respectively.

The formal solution with $f_k = (A_k,\, B_k)$ automatically converges because the components are polynomial and hence satisfy (7.3), and from (7.11) for sufficiently large k,

$$\begin{aligned} \left(\frac{1}{\eta(1+\eta)}\right) + A_k(\eta) &> 0 \qquad \forall 0 < \eta < \delta_2 \\ \left(\frac{1}{\eta(1-\eta)}\right) - B_k(\eta) &> 0 \qquad \forall -\delta_2 < \eta < 0, \end{aligned}$$

and this is the condition we need to guarantee that $\phi' \wedge \phi_2 > 0$ by (7.4.1), hence these solutions generate Joyce metrics g_k on some neighbourhood U_k of $(-\delta_2,\, \delta_2)$. Finally, if

$$A_k(\eta) = \left(\frac{1}{\eta(1+\eta)}\right) + \Delta(\eta) \qquad \eta \in (\delta_1,\, \delta_2),$$

the restriction of the resulting metric is

$$g_J^k|_{\pi^{-1}(\{0\}\times(\delta_1,\,\delta_2))} = \left(\frac{1}{\eta^2} + \Delta(\eta)\right) d\eta^2 + \frac{1}{\left(\frac{1}{\eta^2} + \Delta(\eta)\right)} d\theta_2^2.$$

Then as Δ converges to 0 uniformly, this metric converges to the cusp metric uniformly on $\{0\} \times (\delta_1,\, \delta_2)$.

Performing a similar calculation for B_k on $(-\delta_2,\, -\delta_1)$ shows that the metric also converges uniformly to the cusp metric on $\{0\} \times (-\delta_2,\, -\delta_1)$. □

Then, while it is not possible to find an exact solution because the Joyce potential cannot be analytic, by constructing sequences of polynomials uniformly approximating this potential on appropriate intervals it is possible to construct sequences of Joyce metrics approximating the cusp metric on these intervals.

Chapter 8

Conclusion

We have seen how a toric variety can be constructed from a cone or fan, and how the structure of that fan can tell us about the algebraic structure of the variety. In particular the combinatorial information in the fan can be used to produce blow-ups of the variety and to resolve singularities. We saw that symplectic toric manifolds can be constructed from convex polytopes, that this form gives us an extremely explicit description of the topology of the space and how these two constructions can be related.

We saw that the symmetry of toric 4-manifolds can be used to reduce the self-duality conditions to a pair of first order differential equations, the Joyce equations, for which solutions can be readily found, and how this construction can be used to find a large family of toric Kähler manifolds of zero scalar curvature, and subject to an extra linear constraint, to find self-dual Einstein metrics.

By considering non-compact spaces we were able to use the Joyce construction to find new, complete scalar flat Kähler metrics on manifolds of infinite topological type, generalising the Ricci-flat metrics of Anderson, Kronheimer and LeBrun [26], and many new complete self-dual Einstein metrics on manifolds of infinite topological type, extending the metrics found in [6], and in particular giving spaces with sequences of embedded spheres of unbounded self-intersection.

Examining the Joyce equations directly we were able to find a new family of local solutions and determined when such solutions give rise to new metrics. We then used these new solutions to find a large family of scalar flat Kähler perturbations of the Ooguri-Vafa metric, and to control the boundary behaviour of our Joyce metrics in order to prescribe the metric on the degenerate part of the torus fibration, and to find sequences of scalar flat Kähler metrics on a neighbourhood of the origin in $\mathbb{C}^2$ whose restrictions to $\{(z_1, 0)|\delta_1 < |z_1| < \delta_2\}$ and $\{(0, z_2)|\delta_1 < |z_2| < \delta_2\}$ approach the cusp metric, serving as a model for singular points in fibrations by complex surfaces.

While we can construct many metrics in this way, there are still further possible extensions to be considered. When considering the topological classification of toric 4-manifolds in (3.3.5), we saw that Orlik-Raymond first excluded cases in which interior points in the orbit space had finite stabilisers, $\mathbb{Z}_n \times \mathbb{Z}_m$. It may be possible to find Joyce solutions allowing for these kind of orbits by allowing certain types of singularity at interior points of $\mathcal{H}^2$. This would lead to another broadening of the family of possible solutions, and potentially many new metrics.

Similarly, we have dealt only with orbit spaces homeomorphic to a disc, where other spaces are possible. It may be possible to construct new Joyce metrics by allowing other base spaces and by introducing new boundary components. The local metrics we have constructed here may be useful in approaching this problem, and may serve as models for the local behaviour of such metrics.

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